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Voluntary Participation in the Design of Non-excludable Public Goods Provision Mechanisms

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Public Goods Provision Mechanisms[#]

by

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Abstract

Groves-Ledyard (1977) constructed a mechanism attaining Pareto efficient allocations in the presence of public goods. After this path-breaking paper, many mechanisms have been proposed to attain desirable allocations with public goods. Thus, economists have thought that the free-rider problem is solved, in theory. Our view to this problem is not so optimistic. Rather, we propose fundamental impossibility theorems with public goods. In the previous mechanism design, it was implicitly assumed that every agent must participate in the mechanism that the designer provides. This approach neglects one of the basic features of public goods: non-excludability. We explicitly incorporate non-excludability and then show that it is impossible to construct a mechanism in which every agent has an incentive to participate.

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1. Introduction

Hurwicz (1972), in his path breaking paper, showed that Walrasian mechanism has an incentive problem although many researchers at that time considered that it solves agents' incentive problem. That is, some agents have incentive not to reveal true excess demand functions. Later, Ledyard and Roberts (1974) showed the same problem in public good economies. In other words, it is impossible to design a mechanism that satisfies incentive compatibility where each agent reveals her true utility function or excess demand function as her dominant strategy in private or public good economies.

On the other hand, Groves and Ledyard (1977) designed a Nash implementable mechanism to achieve Pareto efficiency in the presence of public goods. Right after this discovery, Hurwicz (1979) and Walker (1981) designed Nash implementable mechanisms for Lindahl allocations. Hereafter, many mechanisms having nice features have been proposed. Thus, economists have thought that the free-rider problem is solved, in theory.

Our view to this problem is not so optimistic. Rather, we propose fundamental impossibility theorems with public goods. In the previous mechanism design, it was implicitly assumed that every agent must participate in the mechanism that the designer provides. This approach neglects one of the basic features of public goods: non-excludability. In Saijo and Yamato (1999), we explicitly incorporated non-excludability in mechanism design by examining a two-stage game on voluntary participation in a mechanism for providing a non-excludable public good: in the first stage, each agent simultaneously decides whether or not to participate in the mechanism; and in the second stage, after knowing the other agents' participation decisions, the agents who chose participation in the first stage play the mechanism. We fully characterized the equilibrium set of participants in the two-stage game for any second-stage mechanism satisfying symmetry, feasibility, and Pareto efficiency only for participants under any notion of equilibrium in symmetric Cobb-Douglas economies. In particular, we found that there exist economies for which full participation of all agents is not an equilibrium, implying that it is impossible to design reasonable mechanisms in which all agents always have participation incentives. In Saijo and Yamato (1999), however, we made a restrictive assumption that each agent has the same Cobb-Douglas utility function as well as the same endowment.

In this paper, we show the above negative results on participation incentives are robust in the sense that they occur in more general environments. We formulate full participation of all agents as an axiom on a mechanism called the *voluntary participation condition*: each agent always prefers participation to non-participation in the mechanism when all other agents participate in it. We consider any mechanism implementing the Lindahl correspondence, called *Lindhal mechanism*, under any notion of equilibrium,[1] and show that it fails to satisfy the voluntary participation condition in asymmetric Cobb-Douglas utility as well as quasi-linear utility economies in which agents may have different utility functions and endowments. Moreover, we identify the classes of Cobb-Douglas and quasi-linear utility economies for which the voluntary participation condition is satisfied. These classes become smaller and eventually vanish as the number of agents become larger, which can be interpreted as a support for Olson's (1965) conjecture: a public good is less likely provided as the size of a group grows large.

Furthermore, we consider a general version of the voluntary participation condition taking account of various possibilities of each agent's expectation regarding how many agents other than her will participate in the mechanism if she does not participate. In our original voluntary participation condition, each agent is assumed to the most optimistic expectation: all agents other than her will participate in the mechanism if she does not. On the other hand, in the individually rational condition often examined in the literature on mechanism design, each agent has the most pessimistic conjecture: she expects all other *n-1* agents not to participate, and hence no public good is produced. However, an agent might have an intermediate conjecture: her conjecture on the number of other non-participants can take on a whole range of values from 0 to n-1.

Any Lindahl mechanism satisfies the individually rational condition for the most pessimistic expectation, while it fails to meet the voluntary participation condition

[1] In Saijo and Yamato (1999), we examined a class of mechanisms satisfying symmetry, feasibility, and Pareto efficiency only for participants, while we focus on Lindahl mechanisms. However, any mechanism in their class assigns the unique Lindahl equilibrium allocation in each of symmetric Cobb-Douglas utility economies. In this sense, they also limit their attention to Lindahl mechanisms.

for the most optimistic one. Then a natural question arises: how about an intermediate conjecture? We find the critical number of expected non-participants, t^*, to identify whether or not each agent has a participation incentive in symmetric Cobb-Douglas and quasi-linear economies. If each agent expects the number of non-participants to be less than or equal to t^*, then she will not participate; otherwise, she will participate. In particular, as the number of agents, n, grows large, the ratio t^*/n increases, i.e., the possibility for which each agent loses a participation incentive increases. This could be interpreted as another support for Olson's (1965) claim that a public good would be less likely provided as the number of agents increases.

Palfrey and Rosenthal (1984), Moulin (1986), and Dixit and Olson (2000) studied the participation incentive problem in the provision of a public good. In those papers, however, the public good is discrete, while it is continuous in our model. Moreover, the mechanisms studied there are different from ours. Palfrey and Rosenthal (1984) examined voluntary contribution (or provision point) mechanisms with and without a refund to decide whether to produce a discrete public project or not. Contributions are binary and making a fixed contribution can be interpreted as participation in a mechanism. They identified mixed strategy Nash equilibria of the mechanisms. Moulin (1986) used the *no free ride* axiom, requiring each agent have a participation incentive in a mechanism, to characterize the pivotal mechanism in economies with a discrete public good and quasi-linear preferences.

Dixit and Olson (2000) independently considered a two-stage participation game, similar to that in Saijo and Yamato (1999), from the viewpoint of the Coase theorem rather than mechanism design: in the first stage, each agent simultaneously decides whether to participate or not, and in the second stage, those who selected participation play a cooperative game of Coaseian bargaining with no costless enforcement of contracts. They examined a binary public good model like Palfrey and Rosenthal (1984). In particular, they found that the efficient equilibrium outcome of the participation game is not robust when introducing even very small transaction costs. This casts doubt on the validity of Coaseian claims of universal efficiency, which is similar to our negative view in the design of efficient resource allocation mechanisms with participation decisions.

The paper is organized as follows. In Section 2, we explain examples illustrating our basic idea. In Section 3, we introduce notation and definitions. We establish impossibility results on participation incentives in Section 4. In Section 5, we characterize the conditions for which agents lose participation incentive in Cobb-Douglas and quasi-linear utility economies. In Section 6, we consider a generalized version of the voluntary participation condition. In the final section, we make concluding remarks.

2. Examples

Let us consider the following two-agent economies with one private good x and one pure public good y. The public good can be produced from the private good by means of a constant return to scale technology, and let $y = x$ be the production function of the public good. Agent i's consumption bundle is denoted by $(x_i, y) \in \Re_+^2$ where $x_i \in \Re_+$ is the level of private good she consumes on her own, and $y \in \Re_+$ is the level of public good. Each agent has a Cobb-Douglas utility function: $u_i^{\alpha_i}(x_i, y)$ = $\alpha_i \ln x_i + (1-\alpha_i)\ln y$, where $\alpha_i \in (0,1)$ and i = 1,2. Agent i 's initial endowment is given by $(\omega_i, 0)$ for i = 1,2.

Consider any mechanism implementing the Lindahl correspondence (for example, see Hurwicz (1979), Walker (1981), Hurwicz, Maskin, and Postlewaite (1984), and Tian (1990) for Nash implementation[2], and Moore and Repullo (1988) and Varian (1994) for subgame perfect implementation). Suppose that each agent is able to choose whether she participates in the mechanism. Then in order to achieve the desired Lindahl equilibrium allocation by using the mechanism, every agent must choose participation. Therefore, we ask a crucial question of whether each agent always has an incentive to participate in the mechanism. Unfortunately, our answer to this question is negative.

To see why, let $T \subseteq \{1,2\}$ be the set of agents who participate in the mechanism. An equilibrium allocation of the mechanism when the agents in T participate in it is denoted by $((x_i^T)_{i \in T}, y^T)$.[3] If two agents decide to participate in the mechanism, then

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[2] In general, the Lindahl correspondence is not Nash implementable, but it is the same as the *constrained* Lindahl correspondence which is Nash implementable under the present assumptions.

[3] Here we consider a general definition of a mechanism which specifies a strategy set of each participant in T and an outcome function for each $T \subseteq \{1,2\}$.

$(x_1^{\{1,2\}}, x_2^{\{1,2\}}, y^{\{1,2\}})$ should be a Lindahl allocation of the economy consisting of two agents, since the mechanism implements the Lindahl correspondence.[4] It is straightforward to check that there exists a unique Lindahl allocation given by $(x_1^{\{1,2\}}, x_2^{\{1,2\}}, y^{\{1,2\}}) = (\alpha_1\omega_1, \alpha_2\omega_2, \sum_{i=1,2}(1-\alpha_i)\omega_i)$.

Now suppose that some agent *i* does not participate in the mechanism, while the other agent $j \neq i$ does, i.e., $T = \{j\}$. Then $(x_j^{\{j\}}, y^{\{j\}})$ is a unique Lindahl allocation of the economy consisting of only one agent j. It is easy to see that $(x_j^{\{j\}}, y^{\{j\}})$ = $(\alpha_j\omega_j, (1-\alpha_j)\omega_j)$. Notice that non-participant *i* can enjoy her initial endowment, ω_i, as well as the non-excludable public good produced by agent $j \neq i$, $y^{\{j\}}$. On the other hand, she is no longer able to affect the decision on the provision of the public good. Because of this trade-off, it is not obvious whether or not each agent has an incentive to participate in the mechanism. The following condition should be satisfied if each agent has such a participation incentive:

$$u_i^{\alpha_i}(x_i^{\{1,2\}}, y^{\{1,2\}}) \geq u_i^{\alpha_i}(\omega_i, y^{\{j\}}) \quad \text{for } i, j = 1,2, \quad j \neq i,$$

where $u_i^{\alpha_i}$ is any Cobb-Douglas utility function. We call condition (1) the *voluntary participation condition*.[5]

We show that no mechanism implementing the Lindahl correspondence satisfies this condition. This fact can be illustrated by using Kolm's triangle. See Figure 1 in which $(\alpha_1, \alpha_2) = (0.5, 0.7)$ and $(\omega_1, \omega_2) = (10, 20)$. In this economy, agent 1's valuation of the public good is higher than agent 2's, but agent 1 is "poorer" than agent 2. We will see that neither agent has a participation incentive. Point *A* in Figure 1 denotes the Lindahl equilibrium allocation when both agents participate in the mechanism: *A* = $(x_1^{\{1,2\}}, x_2^{\{1,2\}}, y^{\{1,2\}})$ = (5, 14, 11). Point *B* represents the allocation when agent 1 does not

[4] A mechanism is said to implement the Lindahl correspondence if for each set of participants $T \subseteq \{1,2\}$ and each economy consisting of the participants in T, every equilibrium allocation is a Lindahl allocation and every Lindahl allocation is an equilibrium allocation.

[5] The voluntary participation condition is different from the individually rational condition which requires that $u_i^{\alpha_i}(x_i^{\{1,2\}}, y^{\{1,2\}}) \geq u_i^{\alpha_i}(\omega_i, 0)$ for $i = 1,2$. Since $u_i^{\alpha}(\omega_i, y^{\{j\}}) \geq u_i^{\alpha}(\omega_i, 0)$, the voluntary

participate in the mechanism, but agent 2 does: $B = (\omega_1, x_2^{\{2\}}, y^{\{2\}})$ = (10, 14, 6). Since $u_1^{\alpha_1}(x_1^{\{1,2\}}, y^{\{1,2\}}) \approx 2.00367 < u_1^{\alpha_1}(\omega_1, y^{\{2\}}) \approx 2.04717$ for $\alpha_1 = 0.5$, agent 1 prefers Point B to Point A and she does not participate in the mechanism when agent 2 does. The same thing holds for agent 2. In Figure 1, the allocation when agent 2 does not participate in the mechanism, but agent 1 does is represented by Point $C = (x_1^{\{1\}}, \omega_2, y^{\{1\}})$ = (5, 20, 5). Agent 2 prefers Point C to Point A: $u_2^{\alpha_2}(x_2^{\{1,2\}}, y^{\{1,2\}}) \approx 2.56671 < u_2^{\alpha_2}(\omega_2, y^{\{1\}}) \approx 2.57984$ for $\alpha_2 = 0.7$.

Figure 1 is around here.

A similar negative result on voluntary participation in any Lindahl mechanism hold with quasi-linear preferences. Suppose that each agent has a quasi-linear utility function: $u_i^{\beta_i}(x_i, y) = x_i + \beta_i \ln y$, where $\beta_i \in (0, \omega_i)$ (i = 1,2). It is easy to check that a unique Lindahl allocation when both agents participate in the mechanism is given by $(x_1^{\{1,2\}}, x_2^{\{1,2\}}, y^{\{1,2\}}) = (\omega_1 - \beta_1, \omega_2 - \beta_2, \sum_{i=1,2}\beta_i)$ and a unique Lindahl allocation when only one agent j participates in it is $(x_j^{\{j\}}, y^{\{j\}}) = (\omega_j - \beta_j, \beta_j)$. The following voluntary participation condition should be satisfied if each agent has a participation incentive:

$$u_i^{\beta_i}(x_i^{\{1,2\}}, y^{\{1,2\}}) \geq u_i^{\beta_i}(\omega_i, y^{\{j\}}) \quad \text{for } i, j = 1,2, \quad j \neq i,$$

where $u_i^{\beta_i}$ is any quasi-linear utility function.

We will see that no Lindahl mechanism satisfies this condition. Suppose that $(\beta_1, \beta_2) = (2,3)$ and $(\omega_1, \omega_2) = (3,4)$. Then neither agent has a participation incentive. Point A in Figure 2 represents the Lindahl equilibrium allocation when both agents participate: $A = (x_1^{\{1,2\}}, x_2^{\{1,2\}}, y^{\{1,2\}})$ = (1, 1, 5). Point B stands for the allocation when agent 1 does not participate, but agent 2 does: $B = (\omega_1, x_2^{\{2\}}, y^{\{2\}})$ = (3, 1, 3). Since

participation condition is stronger than the individually rational condition. We will discuss differences between the voluntary participation condition and the individually rational condition in more details.

$u_1^{\beta_1}(x_1^{\{1,2\}}, y^{\{1,2\}}) \approx 4.21888 < u_1^{\beta_1}(\omega_1, y^{\{2\}}) \approx 5.19722$ for $\beta_1 = 2$, agent 1 prefers Point B to Point A, in other words, she has no participation incentive when agent 2 participates. The same thing holds for agent 2. In Figure 2, the allocation when agent 2 does not participate, but agent 1 does is denoted by Point *C* = $(x_1^{\{1\}}, \omega_2, y^{\{1\}})$ = (1, 4, 2). Agent 2 prefers Point C to Point A: $u_2^{\beta_2}(x_2^{\{1,2\}}, y^{\{1,2\}}) \approx 5.82831 < u_2^{\beta_2}(\omega_2, y^{\{1\}}) \approx 6.07944$ for $\beta_2 = 3$.

Figure 2 is around here.

3. Notation and Definitions

In the previous section, we see that any Lindahl mechanism fails to satisfy the voluntary participation condition in economies with two agents by looking at certain values of Cobb-Douglas and quasi-linear preference and endowment parameters. We will show similar negative results hold for any number of agents. Also, we will identify classes of preference and endowment parameters for which agents lose participation incentives. In particular, these classes become larger as the number of agents increases.

First of all, we introduce notation and definitions. As in Section 2, there are one private good *x* and one public good *y* with a constant return to scale technology. Let *N* = {1,2,...,*n*} be the set of agents, with generic element *i*. Each agent *i's* preference relation admits a numerical representation $u_i : \Re_+^2 \rightarrow \Re$ which is continuously differentiable, strictly quasi-concave, and strictly monotonic. Let U_i be the class of utility functions admissible for agent *i* and $U \equiv \prod_{i\in I} U_i$. Agent *i*'s initial endowment is denoted by $(\omega_i, 0)$. There is no public good initially. Let Ω_i be the class of private good endowments admissible for agent *i* and $\Omega \equiv \prod_{i\in I} \Omega_i$. An economy is a list of utility functions and endowments of all agents, $e = (u, \omega) = ((u_i)_{i\in N}, (\omega_i)_{i\in N})$ and the class of admissible economies is denoted by $E = U \times \Omega$.

Let an economy $e = (u, \omega) \in E$ be given. Also, let *P*(*N*) be the collection of all no-empty subsets of *N*. Given *T* ∈ *P*(*N*), $e_T = (u_T, \omega_T) = ((u_i)_{i\in T}, (\omega_i)_{i\in T})$ is a sub-economy consisting of agents in T. A *feasible allocation for* e_T is a list $(x_T, y) \equiv ((x_i)_{i \in T}, y) \in \Re_+^{\#T+1}$ such that $\sum_{i \in T}(\omega_i - x_i) = y$. The set of feasible allocations for e_T is denoted by $A(e_T)$.

A *mechanism* is a function Γ that associates with each $T \in P(N)$ a pair $\Gamma(T) = (S^T, g^T)$, where $S^T = \times_{i \in T} S_i^T$ and $g^T : S^T \to \Re^{\#T+1}$. Here S_i^T is the *strategy space of agent* $i \in T$ and g^T is the *outcome function when the agents in T play the mechanism*. Given $g^T(s) = (x_T, y)$, let $g_i^T(s) \equiv (x_i, y)$ for $i \in T$ and $g_y^T(s) = y$. Notice that we assume neither individual feasibility ($g^T(s) \in \Re_+^{\#T+1}$ for all $s \in S^T$) nor balancedness ($g^T(s) \in A(e_T)$ for all $s \in S^T$). Our negative results hold without requiring these conditions.

An *equilibrium correspondence* is a correspondence μ which associates with each mechanism Γ, each economy $e \in E$, and each set of agents $T \in P(N)$, a set of strategy profiles $\mu_\Gamma(e_T) \subseteq S^T$, where $(S^T, g^T) = \Gamma(T)$. Examples of equilibrium correspondences include dominant strategy equilibrium correspondence, Nash equilibrium correspondence, and strong Nash equilibrium correspondence. The set of *μ-equilibrium allocations of* Γ *for* e_T is denoted by $g^T \circ \mu_\Gamma(e_T) \equiv \{(x_T, y) \in \Re^{\#T+1} \mid$ there exists $s \in S^T$ such that $s \in \mu_\Gamma(e_T)$ and $g^T(s) = (x_T, y)\}$, where $(S^T, g^T) = \Gamma(T)$.

Given an economy $e = (u, \omega) \in E$ and a set of agents $T \in P(N)$, a feasible allocation $(x_T, y) \in A(e_T)$ is a *Lindahl allocation for* e_T if there is a price vector $p \in \Re_+^{\#T}$ such that for each agent $i \in T$, $x_i + p_i y = \omega_i$ and $u_i(x_i, y) \geq u_i(x_i', y')$ for any $(x_i', y') \in \Re_+^2$ such that $x_i' + p_i y' \leq \omega_i$. Let $L(e_T)$ be the set of Lindahl allocations for e_T.

Let an equilibrium correspondence μ be given. A *Lindahl mechanism under* μ is a mechanism such that for each economy $e = (u, \omega) \in E$ and each set of agents $T \in P(N)$, $g^T \circ \mu_\Gamma(e_T) = L(e_T)$.

A Lindahl mechanism under μ is a mechanism implementing the Lindahl correspondence in μ-equilibrium, that is, for each set of participants $T \in P(N)$ and each economy consisting of the participants in T, every μ-equilibrium allocation is a Lindahl allocation and every Lindahl allocation is a μ-equilibrium allocation. The above definition of a mechanism implementing the Lindahl correspondence is a generalization of the usual one, in which all agents are supposed to participate, to the case in which voluntary participation is allowed.

4. Impossibility Results on Voluntary Participation

We introduce the following condition on voluntary participation in mechanisms. Let an equilibrium correspondence μ be given.

Definition 1. The mechanism Γ satisfies *voluntary participation for an economy* $e=(u,\omega)$ *under* μ if for all $(x^N, y^N) \in g^N \circ \mu_\Gamma(e_N)$ and all $i \in N$,

$$u_i(x_i^N, y^N) \geq u_i(\omega_i, y_{\min}^{N-\{i\}}),$$

where $y_{\min}^{N-\{i\}} \in \underset{y^{N-\{i\}} \in g_y^{N-\{i\}} \circ \mu_\Gamma(e_{N-\{i\}})}{\arg\min} u_i(\omega_i, y^{N-\{i\}})$. Also, the mechanism Γ satisfies *voluntary participation on the class of economies E under* μ if it satisfies voluntary participation for all economies $e=(u,\omega) \in E$ *under* μ.

Since there is one public good and preferences satisfy monotonicity, $y_{\min}^{N-\{i\}}$ is the minimum equilibrium level of public good when all agents except *i* participate in the mechanism. Consider an agent who decides not to participate in the mechanism. Then she can enjoy the non-excludable public good produced by the other agents without providing any private good, while she cannot affect the decision on the provision of the public good. Voluntary participation requires that no agent can benefit from such a free-riding action. Note that when an agent chooses non-participation, she has a pessimistic view on the outcome of her action: an equilibrium outcome that is most unfavorable for her will occur. Moulin (1986) proposed a similar condition, called the *No Free Ride* axiom, when public goods are discrete and costless, and preferences are quasi-linear.

We will show any Lindahl mechanism fails to satisfy the voluntary participation condition under mild conditions. First of all, consider the class of Cobb-Douglas utility profiles: $U^{CD} = \{(u_i)_{i\in N} \mid \forall i \in N,\ u_i(x_i, y) = u_i^{\alpha_i}(x_i, y) = \alpha_i \ln x_i + (1-\alpha_i)\ln y,$ $\alpha_i \in (0,1)\}$. We have the following negative result on voluntary participation.

Theorem 1. Let μ *be an arbitrary equilibrium correspondence. Suppose that (i) the class of admissible utility profiles* U *includes the class of Cobb-Douglas utility profiles* U^{CD} *and (ii) the class of admissible endowment profiles* Ω *is an arbitrary subset of* $\Re^n_{++}$. *Then any Lindahl mechanism fails to satisfy voluntary participation on* $E = U \times \Omega$ *under* μ.

Proof. Fix any $\omega = (\omega_i)_{i\in N} \in \Re^n_{++}$. For each $(u_i^{\alpha_i})_{i\in N} \in U^{CD}$ and $T \in P(N)$, it is easy to check there exists a unique Lindahl equilibrium allocation for $(u_i^{\alpha_i}, \omega_i)_{i\in T}$, which coincides with a unique μ-equilibrium allocation of the mechanism when agents in T participate in it, given by $(x_i^T,\ y^T) = (\alpha_i\omega_i,\ \sum_{j\in T}(1-\alpha_j)\omega_j)$. Therefore, the difference between agent i's utility level when all agents participate in the mechanism and that when all agents except i participate in it is given by

$$\Delta u_i(\alpha,\omega) \equiv u_i^{\alpha_i}(x_i^N, y^N) - u_i^{\alpha_i}(\omega_i, y^{N-\{i\}}) \tag{4.1}$$

$$= \alpha_i \ln \alpha_i + (1-\alpha_i)\Big\{\ln[(1-\alpha_i)\omega_i + \textstyle\sum_{j\neq i}(1-\alpha_j)\omega_j] - \ln[\sum_{j\neq i}(1-\alpha_j)\omega_j]\Big\},$$

where $(\alpha,\omega) = ((\alpha_j)_{j\in N}, (\omega_j)_{j\in N})$.

We will show that there exist some i and some α such that $\Delta u_i(\alpha,\omega) < 0$, so that the voluntary participation condition is not satisfied. Take $i \in N$ such that $(1-\alpha_i)\omega_i \le (1-\alpha_j)\omega_j$ for all $j \neq i$. Without loss of generality, let $i = 1$. Since $(n-1)(1-\alpha_1)\omega_1 \le \sum_{j\neq 1}(1-\alpha_j)\omega_j$,

$$\Delta u_1(\alpha,\omega) \le \alpha_1 \ln \alpha_1 + (1-\alpha_1)\Big\{\ln[(1/(n-1)+1)\textstyle\sum_{j\neq 1}(1-\alpha_j)\omega_j] - \ln[\sum_{j\neq 1}(1-\alpha_j)\omega_j]\Big\}$$

$$= \alpha_1 \ln \alpha_1 + (1-\alpha_1)\{\ln n - \ln(n-1)\} \equiv f(\alpha_1, n)$$

We prove that the sign of $f(\alpha_1, n)$ is negative when $\alpha_1 = 0.6$ and $n \geq 2$.[6] Note that the function $\ln n - \ln(n-1)$ is decreasing in *n*. Therefore, for $n \geq 2$, $f(0.6, n) \leq f(0.6, 2) \approx$ -0.0292 < 0. This implies that the voluntary participation condition is violated. Q.E.D.

A similar negative result holds for quasi-linear utility economies. Let E^{QL} = $\{((u_i)_{i \in N}, (\omega_i)_{i \in N}) \mid \forall i \in N,\ u_i(x_i, y) = u_i^{\beta_i}(x_i, y) = x_i + \beta_i \ln y,\ \beta_i \in (0, \omega_i)\}$ be the class of quasi-linear utility economies. Here we assume that $\omega_i > \beta_i$ for each *i* to ensure an interior solution.

Theorem 2. Let μ *be an arbitrary equilibrium correspondence and* $e \in E^{QL}$ *be an arbitrary quasi-linear utility economy. Any Lindahl mechanism fails to satisfy voluntary participation for e under* μ.

By Theorem 2, if the class of admissible economies *E* contains a quasi-linear utility economy $e \in E^{QL}$, then any Lindahl mechanism fails to satisfy voluntary participation on *E* under any equilibrium μ.

Proof. Fix any $((u_i^{\beta_i})_{i \in N}, (\omega_i)_{i \in N}) \in E^{QL}$. For each $T \in P(N)$, it is easy to check there exists a unique Lindahl equilibrium allocation at $((u_i^{\beta_i})_{i \in T}, (\omega_i)_{i \in T})$, which coincides with a unique μ-equilibrium allocation of the mechanism when agents in *T* participate in it, given by (x_i^T, y^T) = ($\omega_i - \beta_i$, $\sum_{j \in T} \beta_j$). Therefore, the difference between agent *i*'s utility level when all agents participate in the mechanism and that when all agents except *i* participate in it is given by

$$(4.2) \qquad \Delta u_i(\beta) \equiv u_i^{\beta_i}(x_i^N, y^N) - u_i^{\beta_i}(\omega_i, y^{N-\{i\}}) = \beta_i \left\{-1 + \ln[\beta_i + \textstyle\sum_{j \neq i} \beta_j] - \ln[\textstyle\sum_{j \neq i} \beta_j]\right\},$$

where $\beta = (\beta_j)_{j \in N}$.

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[6] This sign is negative for any $\alpha_1 \in (0.5, 1)$ when $n \geq 2$.

We will show that there exist some i such that $\Delta u_i(\beta) < 0$. Take $i \in N$ such that $\beta_i \le \beta_j$ for all $j \ne i$. Without loss of generality, let $i = 1$. Since $(n-1)\beta_1 \le \sum_{j \ne 1} \beta_j$,

$$\Delta u_1(\beta) \le \beta_1 \left\{-1 + \ln[(1/(n-1)+1)\textstyle\sum_{j\ne 1}\beta_j] - \ln[\sum_{j\ne 1}\beta_j]\right\} = \beta_1\{-1 + \ln n - \ln(n-1)\}.$$

Since the function $\ln n - \ln(n-1)$ is decreasing in n and $\ln 2 - \ln 1 \approx 0.693 < 1$, it follows from the above inequality that $\Delta u_1(\beta) < 0$ for $n \ge 2$. Therefore, the voluntary participation condition is violated . Q.E.D.

5. The Number of Agents and a Participation Incentive: Olson's Conjecture

In the previous section, we show that there exist Cobb-Douglas and quasi-linear utility economies for which some agent fails to have a participation incentive in any Lindahl mechanism, so that the mechanism does not satisfy the voluntary participation condition. However, it is not clear how serious this problem is, that is, how often this negative result occurs. In this section, we check whether or not each agent has a participation incentive in any Lindahl mechanism in any given Cobb-Douglas or quasi-linear utility economy, and identify under which conditions each agent loses a participation incentive. In particular, we are concerned with Olson's (1968) claim that a public good is less likely provided as the size of an economy becomes larger. We present a formal model to confirm this conjecture from the viewpoint of a participation incentive in a mechanism for the provision of a public good. We find that as the size of a Cobb-Douglas or quasi-linear utility economy increases, every agent is less likely to have an incentive to participate in any Lindahl mechanism. The participation incentive disappears in a large size of economy.

First of all, let us consider the case of $U = U^{CD}$, that is, the class of admissible utility profiles is equal to the class of Cobb-Douglas utility profiles. Then an economy is specified by a list of Cobb-Douglas preference paremeters and endowments of all n-type agents, $(\alpha, \omega) \equiv ((\alpha_1, \ldots, \alpha_n), (\omega_1, \ldots, \omega_n))$ such that $\alpha_i \in (0,1)$ and $\omega_i \in (0, \overline{\omega}]$ for all i. Here $\overline{\omega}$ is the upper bound of each endowment. Without loss of generality, we assume that $\overline{\omega} = 1$. Hence, the set of economies is represented by the product of intervals $E^{CD} \equiv (0,1)^n \times (0,1]^n$ endowed with Lebesgue measure λ. Given an economy (α, ω) consisting of n-type agents, we will examine the k-replica of this economy and whether or not each type of agent has an incentive to participate in any Lindahl mechanism. We will find that in a sufficiently large replica of any economy of n-type agents, every type of agent has no participation incentive and hence the measure of the set of economies for the voluntary participation condition is satisfied vanishes.

Take any equilibrium correspondence μ, any Lindahl mechanism Γ under μ, and any Cobb-Douglas economy of n-type agents, $(\alpha,\omega) \in E^{CD}$. Consider the k-replica of this economy in which there are k agents of type (α_i,ω_i) for each $i \in N$. Denote the set of all kn agents in the k-replica economy by kN. Let (x_i^{kN}, y^{kN}) be the consumption bundle each agent of type (α_i,ω_i) receives at the unique Lindahl allocation for kN and $y^{kN-\{i\}}$ be the public good level at the unique Lindahl allocation for $kN-\{i\}$. Also, let

$$\Delta u_i(\alpha,\omega,k) \equiv u_i^{\alpha_i}(x_i^{kN}, y^{kN}) - u_i^{\alpha_i}(\omega_i, y^{kN-\{i\}})$$

be the difference between the utility level of each agent of type (α_i,ω_i) when all agents participate in the mechanism Γ and that when all agents except her, that is, $k-1$ agents of type (α_i,ω_i) as well as $(k-1)(n-1)$ agents of other types participate in Γ in the k-replica economy. If the mechanism Γ satisfies the voluntary participation condition for the k-replica economy, then for any $(\alpha,\omega) \in E^{CD}$ and for any $i \in N$, we must have $\Delta u_i(\alpha,\omega,k) \geq 0$. However, we have the following result:

Theorem 3. Consider an arbitrary equilibrium correspondence μ and an arbitrary Lindahl mechanism Γ under μ. Given any $n \geq 2$, any $(\alpha,\omega) \in E^{CD}$, and any $i \in N$, let $k_i(\alpha,\omega)$ be the largest integer less than or equal to $e^{A(\alpha_i)}(1-\alpha_i)\omega_i / \{(e^{A(\alpha_i)}-1)\sum_{j=1}^{n}(1-\alpha_j)\omega_j\}$, where e is the base of the natural logarithm and $A(\alpha_i) \equiv \alpha_i \ln \alpha_i / (\alpha_i - 1)$. Then $\Delta u_i(\alpha,\omega,k) \geq 0$ for any positive integer $k \leq k_i(\alpha,\omega)$; and $\Delta u_i(\alpha,\omega,k) < 0$ for any positive integer $k > k_i(\alpha,\omega)$.

Proof. It is not hard to check that $(x_i^{kN}, y^{kN}) = (\alpha_i\omega_i, k\sum_{j\in N}(1-\alpha_j)\omega_j)$ and $y^{kN-\{i\}}$ = $(k-1)(1-\alpha_i)\omega_i + k\sum_{j\neq i}(1-\alpha_j)\omega_j$. Therefore,

$\Delta u_i(\alpha,\omega,k)=\alpha_i \ln\alpha_i +$

$$(1-\alpha_i)\Big\{\ln[k(1-\alpha_i)\omega_i + k\textstyle\sum_{j\neq i}(1-\alpha_j)\omega_j)] - \ln[(k-1)(1-\alpha_i)\omega_i + k\textstyle\sum_{j\neq i}(1-\alpha_j)\omega_j)]\Big\}.$$

Let $k_i^*(\alpha,\omega)$ be a value satisfying the equation $\Delta u_i(\alpha,\omega,k_i^*(\alpha,\omega))=0$. This equation can rewritten as

$$\ln\left[\frac{k_i^*(\alpha,\omega)(1-\alpha_i)\omega_i + k_i^*(\alpha,\omega)\sum_{j\neq i}(1-\alpha_j)\omega_j)}{(k_i^*(\alpha,\omega)-1)(1-\alpha_i)\omega_i + k_i^*(\alpha,\omega)\sum_{j\neq i}(1-\alpha_j)\omega_j\}}\right] = A(\alpha_i)\equiv \alpha_i \ln\alpha_i / (\alpha_i - 1).$$

Thus, $k_i^*(\alpha,\omega)=e^{A(\alpha_i)}(1-\alpha_i)\omega_i / \{(e^{A(\alpha_i)}-1)\sum_{j=1}^{n}(1-\alpha_j)\omega_j\}$. Notice that $\Delta u_i(\alpha,\omega,k)$ is strictly decreasing in *k*:

$$\frac{\partial}{\partial k}\Delta u_i(\alpha,\omega,k) = -\frac{(1-\alpha_i)^2\omega_i}{k[(k-1)(1-\alpha_i)\omega_i + k\sum_{j\neq i}(1-\alpha_j)\omega_j]} < 0.$$

Therefore, $\Delta u_i(\alpha,\omega,k) \gtreqless 0$ if and only if $k \lesseqgtr k_i^*(\alpha,\omega)$. This implies the desired result. Q.E.D.

Theorem 3 implies that for any economy $(\alpha,\omega)\in E^{CD}$, no agent has a participation incentive in a sufficiently large replica of the economy. Figure 3 illustrates the result in Theorem 3 when there are n = 2 agents, each agent has the same endowment, $\omega_1=\omega_2$, and the number of replication is k = 1, 2, and 5. As k increases, the region of preference parameters (α_1,α_2) for which neither of two agents has a participation incentive becomes larger and it converges to the entire space $(0,1)\times(0,1)$. In other words, the measure of the set of Cobb-Dogulas utility economies for the voluntary participation condition is satisfied vanishes in a sufficiently large economy.

Figure 3 is around here.

A similar negative result holds for quasi-linear preferences. Suppose that $E=E^{QL}$. Then an economy is specified by a list of quasi-linear preference paramenters and endowments of all n-type agents, $(\beta,\omega)\equiv((\beta_1,\ldots,\beta_n),(\omega_1,\ldots,\omega_n))$ such that

$\beta_i \in (0, \omega_i)$ and $\omega_i \in (0,1]$ for all *i*. In this case, the set of economies is given by $E^{QL} \equiv \{(\beta,\omega) \in (0,1)^n \times (0,1]^n : \beta_i < \omega_i, \forall i\}$. Given an economy (β,ω) consisting of *n*-type agents, we will consider the *k*-replica of this economy in which there are *k* agents of type (β_i, ω_i) for each $i \in N$. For each agent of type (β_i, ω_i), let

$$\Delta u_i(\beta,\omega,k) \equiv u_i^{\beta_i}(x_i^{kN}, y^{kN}) - u_i^{\beta_i}(\omega_i, y^{kN-\{i\}}),$$

where (x_i^{kN}, y^{kN}) is the consumption bundle each agent of type (β_i, ω_i) receives at the unique Lindahl allocation when all agents participate in a Lindahl mechanism and $y^{kN-\{i\}}$ is the public good level produced at the unique Lindahl allocation when all agents except one of agents of type (β_i, ω_i) participate in the mechanism in the *k*-replica economy. We have the following result:

Theorem 4. Consider an arbitrary equilibrium correspondence μ and an arbitrary Lindahl mechanism Γ under μ. Given any $n \geq 2$, any $(\beta,\omega) \in E^{QL}$, and any $i \in N$, $\Delta u_i(\beta,\omega,1) \gtreqless 0$ if and only if $\beta_i \gtreqless (e-1)\sum_{j \neq i}\beta_j$, where e is the base of the natural logarithm; and $\Delta u_i(\beta,\omega,k) < 0$ for any positive integer $k \geq 2$.

Proof. It is not difficult to see that $(x_i^{kN}, y^{kN}) = (\omega_i - \beta_i, k\sum_{j \in N}\beta_j)$ and $y^{kN-\{i\}} = (k-1)\beta_i + k\sum_{j \neq i}\beta_j$. Hence,

$$\Delta u_i(\beta,\omega,k) = \beta_i\left\{-1 + \ln[k\beta_i + k\textstyle\sum_{j\neq i}\beta_j] - \ln[(k-1)\beta_i + k\textstyle\sum_{j \neq i}\beta_j]\right\}.$$

Let $k_i^*(\beta,\omega)$ be a value satisfying the equation $\Delta u_i(\beta,\omega,k_i^*(\beta,\omega)) = 0$. This equation can rewritten as $\ln[\{k_i^*(\beta,\omega)\beta_i + k_i^*(\beta,\omega)\sum_{j\neq i}\beta_j\} / \{(k_i^*(\beta,\omega)-1)\beta_i + k_i^*(\beta,\omega)\sum_{j \neq i}\beta_j\}] = 1$. Therefore, $k_i^*(\beta,\omega) = e\beta_i / \{(e-1)\sum_{j=1}^n \beta_j\}$. Also, note that $\Delta u_i(\beta,\omega,k)$ is strictly decreasing in *k*:

$$\frac{\partial}{\partial k}\Delta u_i(\beta,\omega,k) = -\frac{\beta_i^2}{k[(k-1)\beta_i + k\sum_{j \neq i}\beta_j]} < 0.$$

Hence, $\Delta u_i(\beta,\omega,k) \gtreqless 0$ if and only if $k \lesseqgtr k_i^*(\beta,\omega)$. Let $k=1$. Then $\Delta u_i(\beta,\omega,1) \gtreqless 0$ if and only if $\beta_i \gtreqless (e-1)\sum_{j\neq i}\beta_j$. On the other hand, if $k \geq 2$, then $k > k_i^*(\beta,\omega)$ = $e\beta_i / \{(e-1)\sum_{j=1}^{n}\beta_j\}$, so that $\Delta u_i(\beta,\omega,k) < 0$. Q.E.D.

Figure 4 illustrates the result in Theorem 4 for the case of two agents, n = 2 and no replication, k = 1. In this case, it follows from Theorem 4 that $\Delta u_1(\beta,\omega,1) \gtreqless 0$ if and only if $\beta_2 \lesseqgtr \beta_1/(e-1)$ and $\Delta u_2(\beta,\omega,1) \gtreqless 0$ if and only if $\beta_2 \gtreqless (e-1)\beta_1$. Notice there is no possibility for which $\Delta u_i(\beta,\omega,1) > 0$ holds for all $i \in \{1,2\}$, that is, both agents have a participation incentive. In other words, the measure of the set of economies for the voluntary participation condition is satisfied is zero. No replication of an economy is necessary to obtain this negative result. Moreover, for the just k = 2-replica of any economy $(\beta,\omega) \in E^{QL}$, $\Delta u_i(\beta,\omega,2) < 0$ holds for any $i \in \{1,2\}$, that is, no agent has a participation incentive. This negative conclusion holds for at least two-replica of an arbitrary quasi-linear utility economy. In this sense, the result for quasi-linear utility economies is stronger than that for Cobb-Douglas utility economies.

Figure 4 is around here.

6. A Generalized Voluntary Participation Condition

In the voluntary participation condition defined above, it is implicitly assumed that each agent has the most optimistic conjecture on the number of other agents who will not participate in the mechanism if she does not, that is, she expects no agent other than her to choose non-participation in the mechanism. On the other hand, in the individually rational condition usually discussed in the literature on mechanism design, it is assumed that each agent has the most pessimistic conjecture on that number, that is, she expects all other *n-1* agents to select non-participation, too, so that no public good is produced. However, an agent might have an intermediate conjecture: her conjecture on the number of other non-participants can take on a whole range of values from 0 to n-1.

Taking account of these possible conjectures, we give a general definition of a voluntary participation condition including Definition 1 and the individually rational condition as special cases:

Definition 2. Let t be an integer between 0 and *n-1*. The mechanism Γ satisfies *voluntary participation with respect to t non-participants for an economy* $e=(u,\omega)$ *under* μ if for all $(x^N, y^N) \in g^N \circ \mu_\Gamma(e_N)$, all $i \in N$, and all $T \subseteq N-\{i\}$ such that $\#T = t$,

$$u_i(x_i^N, y^N) \geq u_i(\omega_i, y_{\min}^{N-T\cup\{i\}}),$$

where $y_{\min}^{N-T\cup\{i\}} \in \underset{y^{N-T\cup\{i\}} \in g_y^{N-T\cup\{i\}} \circ \mu_\Gamma(e_{N-T\cup\{i\}})}{\arg\min} u_i(\omega_i, y^{N-T\cup\{i\}})$ and $y_{\min}^{\varnothing} = 0$. Also, the mechanism Γ satisfies *voluntary participation with respect to t non-participants on the class of economies E under* μ if it satisfies voluntary participation with respect to t non-participants for all economies $e=(u,\omega) \in E$ *under* μ.

If $t = 0$, then it is the same as the voluntary participation condition of Definition 1. *If* $t = n$-1, then the condition of Definition 2 is identical to the individually rational condition: a mechanism g satisfies the *individually rational condition on E under* μ if for all $e=(u,\omega) \in E$, all $(x^N, y^N) \in g^N \circ \mu_\Gamma(e_N)$, and all $i \in N$, $u_i(x_i^N, y^N) \geq u_i(\omega_i, 0)$.

In this section, we focus on symmetric economies in which each agent has the same utility function and endowment to clarify conditions to check whether various versions of voluntary participation conditions are satisfied or not. Let U^{SCD} = $\{(u_i)_{i\in N} \mid \forall i \in N,\ u_i(x_i, y) = u_i^\alpha(x_i, y) = \alpha \ln x_i + (1-\alpha)\ln y,\ \alpha \in (0,1)\}$ be the class of symmetric Cobb-Douglas utility profiles. Also, let $E^{SQL} = \{((u_i)_{i\in N}, (\omega_i)_{i\in N}) \mid \forall i \in N,\ u_i(x_i, y) = u_i^\beta(x_i, y) = x_i + \beta \ln y,\ \omega_i = \omega,\ \beta \in (0,\omega)\}$ be the class of symmetric quasi-linear utility economies.

Theorem 5. Let μ *be an arbitrary equilibrium correspondence. Suppose that either (i)* $U = U^{SCD}$ *and* Ω *is an arbitrary subset of* $\{(\omega_i)_{i\in N} \in \Re_+^n \mid \omega_i = \omega > 0, \forall i \in N\}$*, or (ii)*

$E = E^{SQL}$. *For each integer* $n \geq 2$, *let* $t^P(n) \in [0, n-2]$ *be the largest integer less than* $n - 1 - n/e > 0$, *where e is the base of the natural logarithm. Then any Lindahl mechanism fails to satisfy voluntary participation with respect to t non-participants on E under* μ *for any integer* $t \in [0, t^P(n)]$; *and it satisfies voluntary participation with respect to t non-participants on E under* μ *for any integer* $t \in [t^P(n)+1, n-1]$.

In symmetric Cobb-Douglas or quasi-linear utility economies consisting of n agents, each agent has no incentive to participate in any Lindahl mechanism if she expects the number of non-participants other than her to be less than or equal to $t^P(n)$; otherwise, she has a participation incentive. Table 1 illustrates how the value of $t^P(n)$ depends on the number of agents, n. Note that the ratio $t^P(n)/n$ increases as n increases, that is, the possibility for which each agent loses a participation incentive increases as n becomes large. This could be interpreted as another support for Olson's (1965) conjecture that a public good would be less likely provided as the number of agents increases.

n	2	3	4	5	6	7	8	9	10	20	30	40	50	100	500	1000
$t^P(n)$	0	0	1	2	2	3	4	4	5	11	17	24	30	62	315	631

Table 1. The value of $t^P(n)$.

Proof of Theorem 5:

Case-(i): $U = U^{SCD}$ and $\omega_i = \omega > 0$ *for all* $i \in N$. As showed in the proof of Theorem 1, $(x_i^T, y^T) = (\alpha\omega, t(1-\alpha)\omega)$ for $T \in P(N)$, where $\#T = t$. It is easy to check that the difference between each agent *i's* utility level when all agents participate in the mechanism and that when the agents in the group $N - T \cup \{i\}$ participate in it is given by

$$u_i^{\alpha}(x_i^N, y^N) - u_i^{\alpha}(\omega_i, y^{N-T\cup\{i\}}) = \alpha \ln \alpha + (1-\alpha)[\ln n - \ln(n-1-t)] \equiv f(\alpha, n, t).$$

We will identify the sign of $f(\alpha,n,t)$. For each $\alpha \in (0,1)$ and each integer $n \geq 2$, let $t^*(\alpha,n)$ be a value satisfying the equation $f(\alpha,n,t^*(\alpha,n)) = 0$. We can rewrite this equation as $\ln[n/(n-1-t^*(\alpha,n))] = k(\alpha) \equiv \alpha \ln\alpha/(\alpha-1) > 0$. Hence, $t^*(\alpha,n) = n-1-n/e^{k(\alpha)}$. The following properties on the functions $k(\alpha)$ and $t^*(\alpha,n)$ will be useful below:

Claim 1. (a) $k(\alpha)$ is strictly increasing in $\alpha \in (0,1)$; (b) $\lim_{\alpha\to 1} k(\alpha) = 1$;

(c) $t^*(\alpha,n) < n-1-n/e$ for any $\alpha \in (0,1)$; and (d) $f(\alpha,n,t) \gtreqless 0$ if and only if $t \gtreqless t^*(\alpha,n)$.

Proof of Claim 1: (a) Note that $dk(\alpha)/d\alpha = (\alpha-1-\ln\alpha)/(\alpha-1)^2$. Since $(\alpha-1)^2 > 0$, it remains to prove that $A(\alpha) \equiv \alpha - 1 - \ln\alpha > 0$. It is easy to check that $dA(\alpha)/d\alpha < 0$ if $\alpha \in (0,1)$, $dA(\alpha)/d\alpha = 0$ if $\alpha = 1$, and $A(1) = 0$. Therefore, $A(\alpha) > 0$ for $\alpha \in (0,1)$.

(b) By L'Hôpital's rule, $\lim_{\alpha\to 1} k(\alpha) = \lim_{\alpha\to 1}[(\ln\alpha + 1)/1] = 1$.

(c) By Claim 1-(a), the function $t^*(\alpha,n)$ is strictly increasing in $\alpha \in (0,1)$. By Claim 1-(b), $\lim_{\alpha\to 1} t^*(\alpha,n) = n-1-n/e$. These facts imply the desired result.

(d) The desired result immediately follows from the fact that the function $\ln[n/(n-1-t)]$ is strictly increasing in *t*. Q.E.D.

Take any integer $n \geq 2$. Let $t^P(n)$ be the largest integer less than $n-1-n/e > 0$. We will prove that there is $\alpha^* \in (0,1)$ such that $f(\alpha^*,n,t) < 0$ for any integer $t \in [0, t^P(n)]$, that is, voluntary participation with respect to *t* non-participants is not satisfied. By the definition of $t^P(n)$, $t^P(n) < n-1-n/e$. Also, the function $t^*(\alpha,n) = n-1-n/e^{k(\alpha)}$ is continuous in α and by Claim 1-(a), it is increasing in $\alpha \in (0,1)$. Further, by Claim 1-(b), $\lim_{\alpha\to 1} t^*(\alpha,n) = n-1-n/e$. These facts together imply that for some $\alpha^* \in (0,1)$, which is sufficiently close to 1, $t^P(n) < t^*(\alpha^*, n) = n - 1 - n / e^{k(\alpha^*)} < n - 1 - n / e$. By Claim 1-(d), $f(\alpha^*, n, t) < 0$ if $t \le t^P(n) < t^*(\alpha^*, n)$.

Next we will show that for any integer $t \in [t^P(n) + 1, n - 1]$ and any $\alpha \in (0,1)$, $f(\alpha, n, t) > 0$, that is, voluntary participation with respect to t non-participants is satisfied. By Claim 1-(c), $t^*(\alpha, n) < n - 1 - n / e$ for any $\alpha \in (0,1)$. Also, by the definition of $t^P(n)$, $n - 1 - n / e < t^P(n) + 1$. Therefore, $t^*(\alpha, n) < n - 1 - n / e < t^P(n) + 1 \le t$ for any $t \in [t^P(n) + 1, n - 1]$. Since $t^*(\alpha, n) < t$, it follows from Claim 1-(d) that $f(\alpha, n, t) > 0$. Therefore, we have the desired result.

Case-(ii): $E = E^{SQL}$. As showed in the proof of Theorem 2, $(x_i^T, y^T) = (\omega - \beta, t\beta)$ for $T \in P(N)$, where $\#T = t$. It is easy to see that the difference between each agent's i's utility level when all agents participate in the mechanism and that when all agents in the group $N - T \cup \{i\}$ participate in it is given by

$$\Delta u_i^\beta(T) \equiv u_i^\beta(x_i^N, y^N) - u_i^\beta(\omega_i, y^{N-T\cup\{i\}}) = \beta\{-1 + \ln n - \ln(n - 1 - t)\} \equiv \beta \cdot g(n,t).$$

Let $t^*(n)$ be a value satisfying the equation $g(n,t) = 0$. Then $t^*(n) = n - 1 - n / e$.

Since $g(n,t) \gtreqless 0$ if and only if $t \gtreqless t^*(n)$, we have the desired result. Q.E.D.

7. Concluding Remarks

We see that the solutions to the free-rider problem, which have been proposed in mechanism design theory, are not necessary solutions to the free-rider problem when participation in mechanisms is voluntary. It is quite difficult or impossible to design Lindahl mechanisms with voluntary participation: any Lindhal mechanism fails to satisfy the voluntary participation condition in Cobb-Douglas and quasi-linear economies. Also, it is not hard to check that a similar negative result hold for the voluntary contribution mechanism that does not satisfy Pareto efficiency only for participants when the equilibrium concept is Nash equilibrium.

Cason, Saijo, and Yamato (2002) and Cason, Saijo, Yamato, and Yokotani (2004) observed that cooperation has emerged though spiteful behavior in their experiments on the voluntary contribution mechanism with voluntary participation. Our theory in this paper suggests that no cooperation will emerge. Reconciling theoretical results to experimental results is an open area of our future research.

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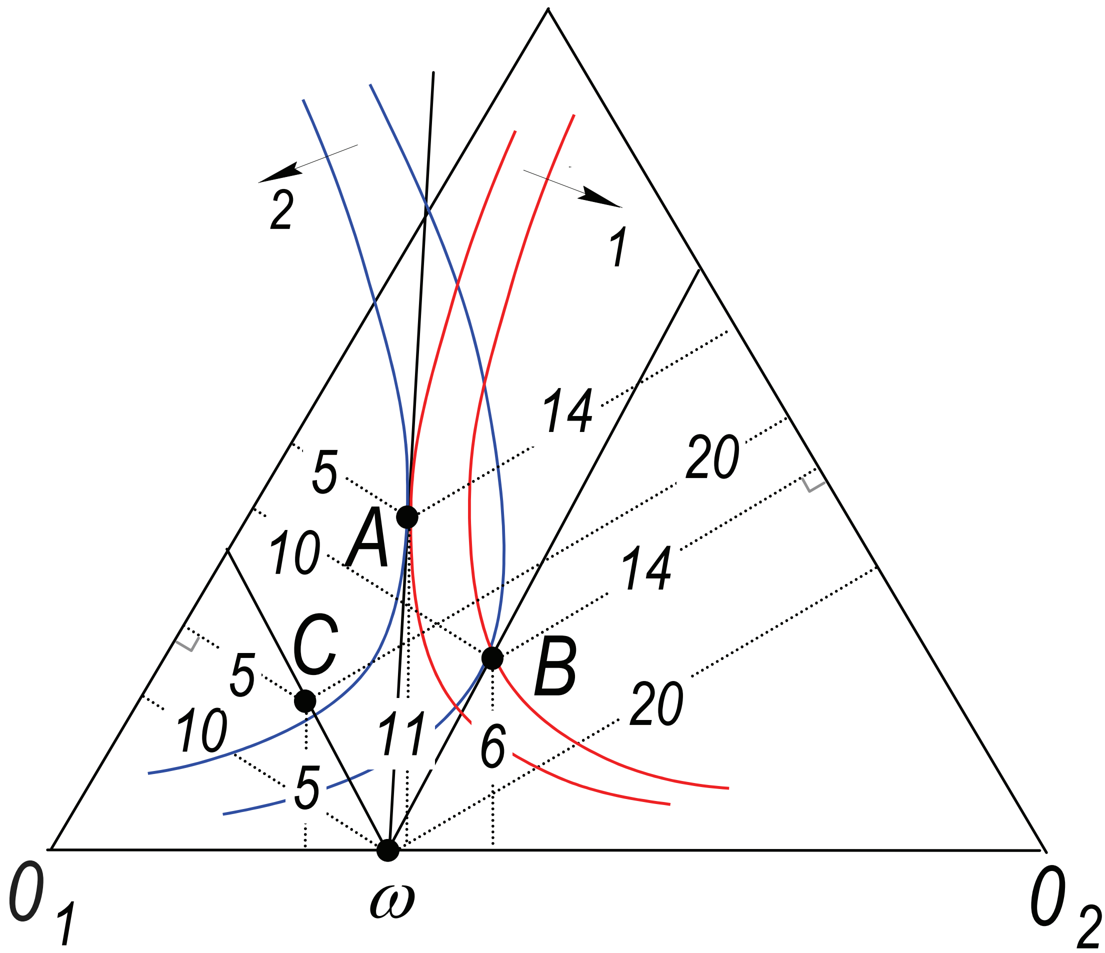
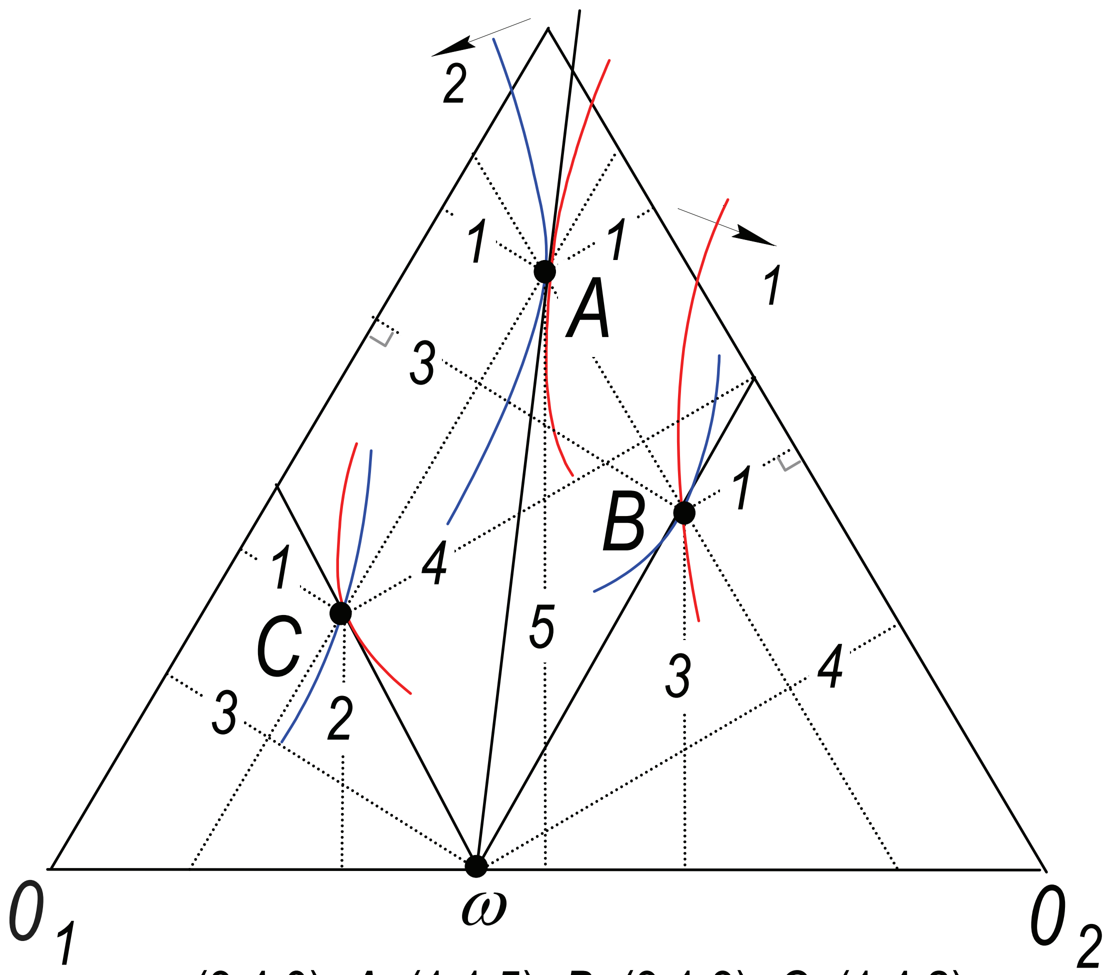


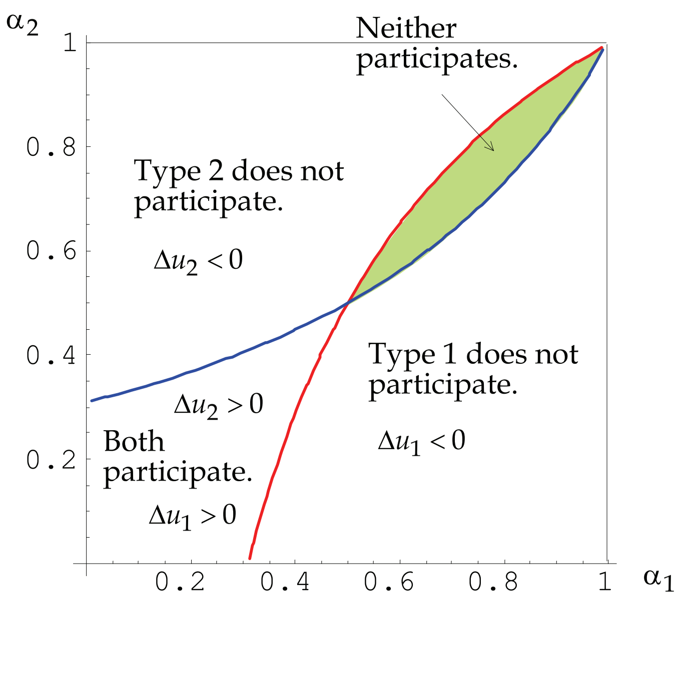
ω =(10,20,0), A=(5,14,11), B=(10,14,6), C=(5, 20, 5)

Figure 1. No Lindahl mechanism satisfies the voluntary participation condition when preferences are Cobb-Douglas.

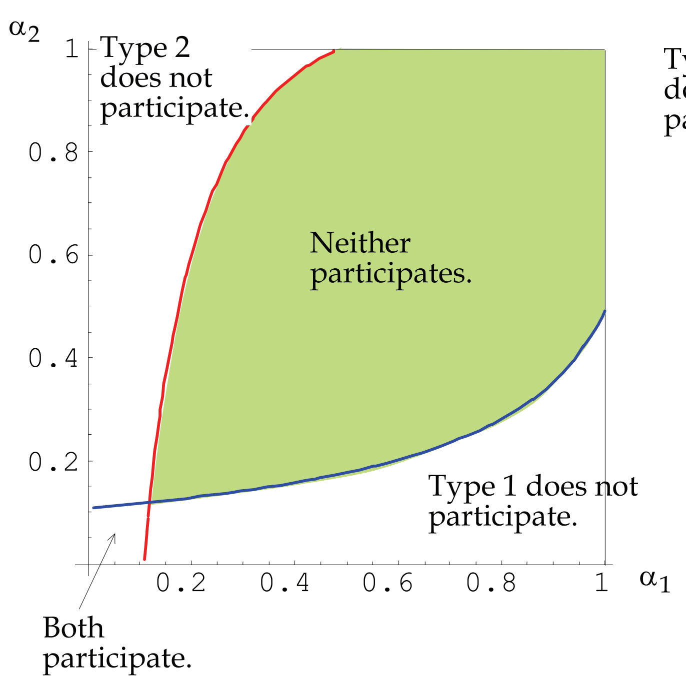


ω=(3,4,0), A=(1,1,5), B=(3,1,3), C=(1,4,2)

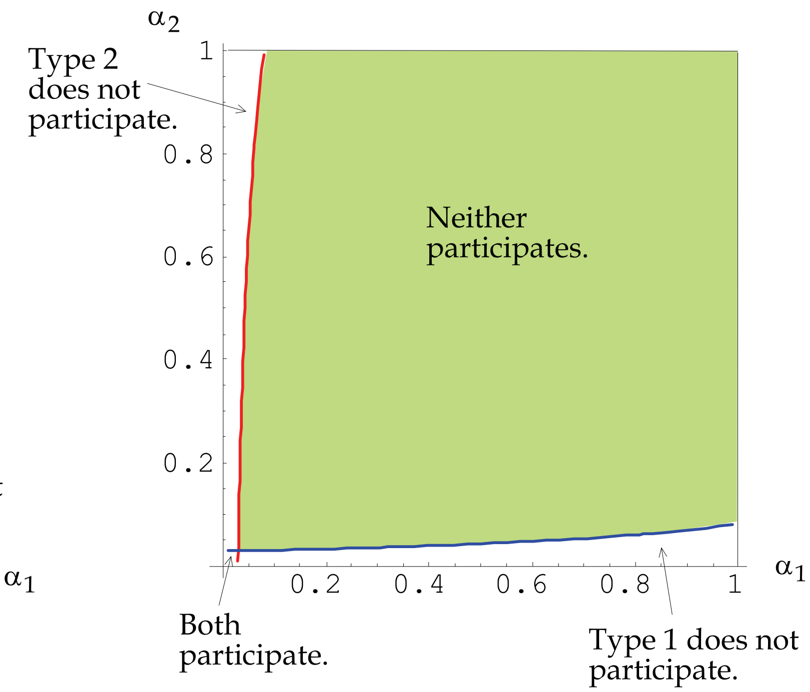
Figure 2. No Lindahl mechanism satisfies the voluntary participation condition when preferences are quasi-linear.



(a) The $k = 1$-replica.



(b) The $k = 2$-replica.



(c) The $k = 5$-replica.

Figure 3. Participation incentives in two-agent Cobb-Dogulas utiltiy economies with symmetric endowments.

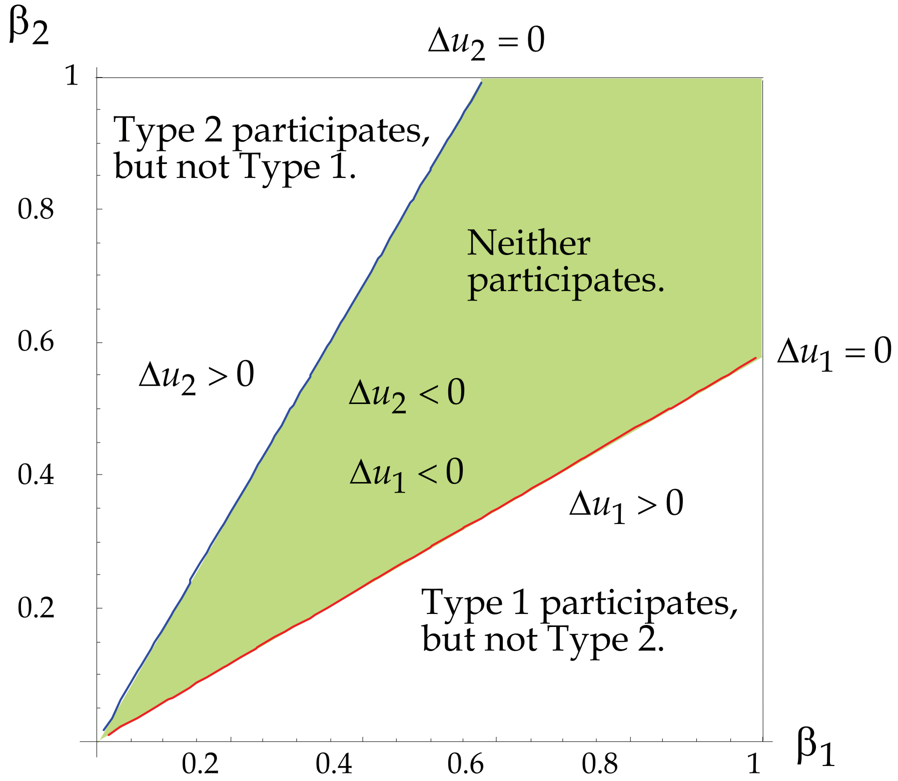


Figure 4. Participation incentives in two-agent quasi-linear economies: the $k = 1$-replica case.