PRODUCTION THEORY WITH FRICTION

KOJI YOKOTA

ABSTRACT. A general production theory with friction is provided in which a firm employs multiple workers facing search friction representable by a vacancy cost function robust against any small perturbations. With such a vacancy cost function, the path outside of unbounded steady state is not only significant as transition but actually essential in the sense that the unbounded steady state is generally never reached. Analysis of out of unbounded steady state requires various novel tools which are also represented in this paper: employment of integral equations to obtain wages, optimization on state constraints and a particular change of coordinates which is generally applicable in solving this type of problems. Moreover, the market outcome of such a model, even though the hiring cost can be infinitesimally small for a given amount of job posting, is that a generalized effective demand principle must be effective for a market equilibrium to exist. This result comes from the fact that the marginal profit value of labor is always strictly positive, which results in persistent excess demand in the labor market and excess supply in the goods market, whereas the existence of search friction prohibits adjustment of employment toward infinity *by optimality*. It should be noted that any price or wage rigidity is not assumed to obtain this result.

1. INTRODUCTION

The present paper studies optimal employment policy of a firm under the presence of a convex vacancy cost function in the labor market, and its implication on employment distribution and market equilibrium. It is a generalization of a search model from one-to-one matching to one-to-many matching. A convex vacancy cost function ——a function which literally relates number of job vacancy posting to cost—— is chosen since it is the only class the derivative of which is monotone, which is robust against any small perturbations and which does not diverge in equilibrium. Requirement of robustness on a vacancy cost function would be natural. There is no logically strong, *a priori* reason that a vacancy cost function must have a particular functional form. Any results derived from an unrobust assumption in this respect are unlikely to hold in reality.

It will be shown that marginal wage cost value determined by bargaining is *always* strictly smaller than the marginal production value of labor as far as employment is below an unbounded steady state. On the other hand, the optimal employment policy under a convex vacancy cost function does not allow the path to jump to an unbounded steady state. Therefore, a firm is willing to accommodate all the demand directed to it ——let us call it potential demand to distinguish it from effective demand—— unless it is strictly larger than the unbounded steady state. For an economy to be directed to an unbounded steady state, economic agents must share a dynamically

This is a preliminary paper. It was previously circulated with the title "Employment Policy of a Firm Under the Presence of Search Friction". The author would greatly appreciate useful comments by Haruo Imai, Ryoichi Imai, Akira Kawaguchi, Keiichiro Kobayashi, Masataka Kudo, Yoshiaki Omori, Tsunao Okumura, Soichi Ota, Fumio Otake, Toyotaka Sakai, Masaru Sasaki, Tadashi Sekiguchi, Akihisa Shibata, Toshiaki Tachibanaki and audience of Kansai Labor Seminar, Summer Workshop on Economic Theory and seminars at Keio University, Kyoto Institute for Economic Research, Osaka University and Yokohama National University.
Otaru University of Commerce, Midori 3 – 5 – 21, Otaru, 047 – 8501, Japan. +81 – (0)134 – 27 – 5324.

persistent common sense that the economy ultimately reaches to an unbounded steady state. However, until the economy reaches to the unbounded steady state, production and income continue to be strictly below potential demand, and thus this view solely depends on optimistic expectation on future, which is not guaranteed to hold under competitive environment. Here, we have a generalized principle of effective demand to dynamics. Namely, national income is determined by the production achievable by the current level of employment and its growth is bounded by the employment technology *and* the growth of potential demand which reflects the state of expectation of economic agents. The unbounded steady state is equivalent to the one sometimes had been called a "state in the long run", thus it might be safely said that we are all dead before such a state is visited unless our expectation is coordinated.

The model presented in this paper is a generalization of a Mortensen-Pissarides model which assumes that production is undertaken by a pair of a worker and a job. Such a model can be interpreted that it assumes a "firm" employing multiple workers is decomposable to independent units of jobs. Since it assumes cost of vacancy posting is constant, it actually hypothesizes a linear vacancy cost function. It assumes that the size of employment in the economy is determined by the entry condition that the value of vacancy equals zero. However, under the assumption that each production pair always *successfully* earns constant income, the size of employment this condition requires can be "huge", possibly exceeding our population. Potential entrepreneurs cease job posting because labor market tightness makes waiting time for the arrival of a worker too long compared to vacancy cost. Although it is not necessarily clear what kind of costs are included in vacancy cost, from the view point that cost to post information to various media is negligible in this IT society, and also from the fact that cost required for selection of applicants should decrease as market condition becomes tight because applicants decreases, the VU ratio must be extremely high in equilibrium. The waiting time must be long enough to make entrepreneurs give up the *existing* production opportunity.

Section 2 summarizes the structure of the model. Section 3 studies how the value of unemployment which operates as a threat point in wage bargaining is determined. Section 4 studies the outcome of wage bargaining. It is shown that integral equation is useful to solve non-stationary value functions. Section 5 studies the firm's optimal employment behavior. Section 6 rationalizes the assumptions made in Section 3 are actually consistent with the whole model. Section 7 analyzes the behavior on the demand constraint when the constraint is stationary. Section 8 provides some conclusional remarks.

2. The Model

2.1. **Firm.** A competitive firm under the presence of search friction in the labor market is considered. Labor is known to be heterogeneous so that optimal search behavior is not trivial. It uses multiple workers of potentially multiple types. In general, we can presume two kinds of "types", declarable and non-declarable. Declarable types are those which can be prescribed as hiring requirements, such as possession of driver's license and academic background. Non-declarable types are those which cannot be documented such as personality and suitability to particular corporate culture, therefore it cannot be observed until, for example, a firm holds an interview. Therefore, it would be natural to assume non-declarable types are matching-specific. The types of workers are generally combination of these declarable and non-declarable types. Let (i, j) denote a bundle of labor types in which declarable type is i and non-declarable type is j. Different types are clearly distinguished from each other. For simplicity, it is assumed that workers cannot choose their declarable types, abstracting the effect of the choice of education and training. Any workers of the same type are perfectly homogeneous for the firm. The production function of a firm is $f(\mathbf{l})$ where $\mathbf{l} = (\mathbf{l}_1, \dots, \mathbf{l}_L)$ and $\mathbf{l}_i = (l_{i1}, \dots, l_{iM_i})$ are vectors of the number of employed workers such that l_{ij} is that of (i, j)-type labor, $\partial f/\partial l_{ij} > 0$, $\partial^2 f/\partial l_{ij}^2 < 0$ and $\partial^2 f/\partial f_{ij}\partial f_{i'j'} > 0$ where $i' \neq i$ and $j' \neq j$. We also assume Inada condition around the origin: $\partial f/\partial l_{ij} \to +\infty$ as $l_{ij} \to 0$.

A firm decides how much internal resources to spend on recruiting labor force in the labor market. After the choice of the level of recruiting activities $\mathbf{m} = (m_1, \dots, m_L)$ for each declarable type, it would observe a variety of applicants to arrive stochastically.[1] A matching session proceeds in a way that firms post job advertisement first and workers apply to a preferred job. Such a matching mechanism is a natural equilibrium of such environment that, while many characteristics of a firm is declarable, that of a worker is non-declarable, as in Yokota (2004). It results in the same outcome as random matching, so that the probability that a firm receives applications per job posting is a decreasing function of VU ratio θ_i in the labor market of declarable type i and denoted by $\psi(\theta_i) \in \mathbb{R}_+$.[2] On the other hand, non-declarable types of workers are distributed in a way that share of non-declarable type j is $g_{ij} \in (0, 1)$ for declarable type i such that $\sum_j g_{ij} = 1$. Therefore, if a firm exerts recruiting effort m_i on declarable type i in the labor market, it will receive $g_{ij}\psi(\theta_i)m_i$ applications from type (i, j) worker.

On the other hand, a pair of a firm and a worker breaks up for external reasons to the firm. Suppose that a separation event occurs with time-variant Poisson arrival with parameter $\sigma_{ij}(t) > 0$ at time t for $i = 1, \dots, L$ and $j = 1, \dots, M_i$.[3] In addition, a firm can dismiss type (i, j) workers by x_{ij}. Note that it can specify which type to dismiss as it already knows non-declarable characteristics of workers. Then, a firm can control the time derivative of type i employment with control variables m_i and x_{ij} so that

$$\dot{l}_{ij} = g_{ij}\,\psi(\theta_i)\,m_i - \sigma_{ij} l_{ij} - x_{ij} \qquad \forall i = 1, \dots, L. \tag{2.1}$$

For simplicity, the notation $\phi_{ij}(t) := g_{ij}(t)\,\psi(\theta_i(t))$ is sometimes used.

Job posting is assumed to be costly. Smith (1999) assumed a linear vacancy cost function, so that a firm employs all necessary workers to reach to the steady state in the first period, and then it maintains the steady state forever. With this carefully arranged setup, adjustment process to the steady state is virtually abstracted. However, as described in Section 1, its assumption is not robust against small perturbations to the functional form. With

[1]By equation (2.1), m_i is directly related to the increase of labor force. It is labeled as "level of recruiting activities" instead of "number of job vacancies" to abstract the strategic behavior to announce more job posts than actually wanted.

[2]This is a special case of Yokota (2004) when the threshold that a firm declines an applicant becomes zero to conform the current problem. Here, a firm can employ multiple job-seekers and the choice which applicants to decline is detached to the choice of x_i.

[3]It may be more natural to assume that the quality of a match gradually turns out on the job as in Jovanovic (1979). However, we abstract internal working of separation.

a minimalist principle, we are induced to assume a convex vacancy cost function, since its derivative is simply monotone and its equilibrium outcome is implied not to diverge. It would be also natural from the viewpoint that a vacancy cost function should be regarded as an adjustment cost function. In practical application, the cost may be interpreted as including the cost for orientation, training and deterioration of productivity that arises from on-the-job training and inexperience of new workers, as well as the cost necessary for actual recruiting. Modifying the functional form of the vacancy cost function in this way surely complicates the analysis than that of Smith (1999), but it also should be pointed out that such a model based on robust assumptions brings significantly different macroeconomic implications. We denote the vacancy cost function for declarable type $i = 1, \ldots, L$ by $\kappa_i(m_i, t) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ where $\kappa_i \geq 0$, $\partial\kappa_i/\partial m_i \geq 0$, $\partial^2\kappa/\partial m_i^2 > 0$, $\kappa_i(0, t) = \partial\kappa_i/\partial m_i(0) = 0$, and for simple notation, the second argument will be omitted from now on: $\kappa_i(\cdot) := \kappa_i(\cdot, t)$. In this way, κ_i is allowed to depend on θ_i. A firm pays wages to each type of workers. Under the presence of search friction, wage payment is determined in a way that workers and a firm share pseudo-rent which an already-formed group possesses. As shown later, wage rate is a function of employment. A firm decides the amount of employment knowing the wage schedule it faces. We denote the wage function of type (i, j) by $w_{ij}(\mathbf{l})$. Normalizing the price of output to one, the instantaneous profit of a firm π is given by

$$\pi = f(\mathbf{l}) - w(\mathbf{l}) \cdot \mathbf{l} - \sum_{i=1}^{L} \kappa_i(m_i).$$

Again, for simple notation, it is sometimes used $c_{ij}(\mathbf{l}) := w_{ij}(\mathbf{l})\, l_{ij}$ for any (i, j).

Before examining the intertemporal optimal behavior of a firm, we need to examine the wage function. Bargaining between a firm and workers is done by sharing pseudo-rent represented by the present value of matched status discounting all possible future status. After deriving the bargaining rule in terms of value functions in Section 3, we derive the wage function in Section 4.

2.2. **Workers.** Workers are in either state, employed or unemployed. An unemployed worker of type $(i, \cdot)$ at time t receives instantaneous unemployment benefit $b_i(t)$. An employed worker of type (i, j) will be paid instantaneous wage $w_{ij}(t)$. We denote by $U_i(t)$ the value of unemployment of type $(i, \cdot)$ at time t and denote by $E_{ij}(t)$ the value of employment of a type (i, j) worker at time t. Matching sessions between job-seekers and vacancies open at any moment. Matching probability of an unemployed worker is $\mu_i(t)$. A matching session is time-consuming, and its length is random. While an agent joins in it, he cannot attend any other sessions that will be held at the same time.[4] If an unemployed worker of type i is successfully matched at time t and it turns out that his undeclarable type is j, then he moves to the state of employment of type (i, j). Namely, he receives the "value" of employment of type (i, j) at time t, $E_{ij}(t)$. We assume that a worker is risk neutral. Then, the Bellman equation for the value of

[4]If matching sessions open continuously and end instantaneously, all agents in the labor market will be matched immediately almost surely. The best analogy for matching in a continuous context would be to regard it as two "drains" which has a limited and the same capacity attached to different tanks of "workers" and "firms".

unemployment becomes

$$r(t)\,U_i(t) = b_i\,(t) + \mu_i(t)\mathsf{E}_j\left[E_{ij}(t) - U_i(t)\right] + \dot{U}_i(t) \tag{2.2}$$

where r is interest rate and $\mathsf{E}_\mathsf{j}\left[E_{ij}(t) - U_i(t)\right] := \sum_{j=1}^{M_i} g_{ij}E_{ij}(t) - U_i(t)$. Similarly, the value of employment of type i at time t is given by

$$r(t)\,E_{ij}(t) = w_{ij}(t) - \sigma_{ij}\left[E_{ij}(t) - U_i(t)\right] + \dot{E}_{ij}(t) \tag{2.3}$$

for all $i = 1, 2$.

3. Bargaining over Coalitional Rent

When there exists friction in the labor market, pseudo-rent arises in an existing firm-workers group. The rent comes from the fact that any firms or workers who have not formed a group yet must enter a costly process of search. It makes a room for bargaining over the rent between a firm and workers who have already formed a group. Therefore, production should be regarded as coalitionally undertaken by the going concern and the present and future employees and distributed by bargaining. It would be reasonable to regard the bargaining as based on the contribution of each participant, which leads us to focus on nucleolus and Shapley value as bargaining outcome.[5] Coalition of a firm and each type of workers with measure l_{ij} will get an intertemporal payoff F which is the value of

$$F(\mathbf{l}) = \int_t^\infty \left[f(\mathbf{l}(\xi)) - \sum_{i=1}^{L} \kappa_i(m_i(\xi)) \right] e^{-R(t,\xi)} d\xi$$

where $R(t,\xi) := \int_t^\xi r(\tau)\,d\tau$, and which is obtained when *the firm follows its own optimal policy*. Then, bargaining is made among continuously many players. For the moment, we proceed with the argument in a general setup which function F should satisfy, to facilitate future extension. The bargaining among continuously many players should be obtained as a limit of the games with finite players.

Let Ω be a set of all players. Let $\Upsilon := \{\Upsilon_1, \ldots, \Upsilon_L\}$ where $\Upsilon_i := \{1, \ldots, M_i\}$ $(i = 1, \ldots, L)$ for given $M_i \in \mathbb{N} \cup \{\infty\}$, and $\Psi_{ij} := \{1, 2, \ldots, N_{ij}\}$ where $N_{ij} \in \mathbb{N} \cup \{\infty\}$ is given for any $(i, j) \in \Upsilon$. Define a vector $\mathbf{N} := (\mathbf{N}_1, \ldots, \mathbf{N}_L)$ where $\mathbf{N}_i := (N_{i1}, \ldots, N_{iM_i})$. Structure of players is that there exist partition $S_0, \{S_{ij}\}_{(i,j)\in\Upsilon} \subset \Omega$ such that $\Omega = S_0 + \sum_{(i,j)\in\Upsilon} S_{ij}$ where $S_0 = \{s_0(1)\}$ and $S_{ij} = \{s_{ij}(1), s_{ij}(2), \ldots s_{ij}(N_{ij})\}$. $s_{ij}(n)$ has measure dl_{ij} for all $(i, j) \in \Upsilon$ and $n \in \Psi_{ij}$. There also exists a fixed vector $(l_{11}, \ldots, l_{LM_L})$ such that $N_{ij}dl_{ij} = l_{ij}$ for all N_{ij}, dl_{ij} and $(i, j) \in \Upsilon$. Let $X_h : 2^\Omega \to \mathbb{R}^{\|2^\Omega\|}$ be a multi-valued (pre)imputation function of h-th game where $J := X_h\left(\{s_0(1)\}\right)$ is a payoff of player $s_0(1)$ and $E_{ij}(n) := X_h\left(\left\{s_{ij}(n)\right\}\right)$ is that of player $s_{ij}(n)$. For given $\mathbf{N}$, we define a series of reduced coalitional games as follows.

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[5] Pissarides (1985) assumed that, in the case that production is undertaken by a pair of a firm and a worker, they divide the rent by a Nash bargaining solution. It would have been an option to generalize it to a multiple-worker environment adopting n-player Nash equilibrium as an equilibrium notion. However, as the present model contains significant asymmetry between a firm and workers, it seems more natural to take coalitional rationality into account.

Definition 3.1. Define a series of coalitional games $\{\eth_{\mathbf{N}}(\mathcal{S}_h, X_h)\}_{h\in\mathbb{N}}$ for given $\mathbf{N}$ when coalition family $\mathcal{S} \subset 2^{\Omega}$ is given with the following properties.

(1) The set of players is $\Omega \cap \left\{s_{ij}(n) : X_h\left(\left\{s_{ij}(n)\right\}\right) = 0\right\}$ and feasible coalition is given by $\mathcal{S}_h$.

(2) The characteristic function v has the property that, for any coalition $A \in \mathcal{S}_h$,

$$v(A) = \begin{cases} F\left(l'_{11}, \ldots, l'_{LM_L}\right) - \sum_{s_{ij}(n)\in A} X_h\left(\left\{s_{ij}(n)\right\}\right) & \text{if } c \in A \text{ and } \forall(i,j), \exists n,\ s_{ij}(n) \in A \\ -X_h(S_0) & \text{if } \{c\} = A \\ \sum_i U_i \left\|A \cap \left(\cup_j S_{ij}\right)\right\| - \sum_{s_{ij}(n)\in A} X_h\left(\left\{s_{ij}(n)\right\}\right) & \text{if } c \notin A \text{ or } \exists(i,j), \forall n,\ s_{ij}(n) \notin A \end{cases}$$

where $l'_{ij} = \left\|A \cap S_{ij}\right\| dl_{ij}$ for all (i, j), F is an increasing function and concave in terms of each variable. F is common for all $S \subset \Omega$ and U_i common for all $\cup_j S_{ij}$.[6]

(3) The transition of games is given by the following rule.

  (a) $\mathcal{S}_0 = 2^{\Omega}$ and $X_0(A) = 0$ for all $A \in \mathcal{S}_0$.

  (b) Find $\varepsilon_i = \min\max\{v(S) - X(S)\}$. Let $\mathcal{T}_h = \{S \in \mathcal{S}_h : X(S) = v(S) - \varepsilon_h\}$ for $h = 1, 2, \ldots$. Then, the transition of games is given by $\mathcal{S}_{h+1} = \mathcal{S}_h \setminus \mathcal{T}_h$ and

$$X_{h+1}(S) = \begin{cases} v(S) - \varepsilon_h & \text{if } S \in \mathcal{T}_h \\ 0 & \text{otherwise} \end{cases}$$

$\eth_{\mathbf{N}}(2^{\Omega}, X_0)$ defines the game itself for given $\mathbf{N}$. Our objective here is to obtain a bargaining solution for $\eth_{\boldsymbol{\infty}}(2^{\Omega}, X_0)$ where $\boldsymbol{\infty} := (\infty, \ldots, \infty) \in \mathbb{N}^M$. The game is superadditive and, in addition, player $s_0(1)$ has discrete influence on coalitional payoff. In that sense, player c is a "mass" player. Obviously, $C = \{s_0(1)\}$ refers to a firm and S_{ij} to a set of type (i, j) workers in our context. Note that the properties of F is sufficient to hold at the current level of employment *only from below*.

Lemma 3.2. *Core is nonempty if* $\partial F/\partial l_{ij} \geq U_i$ *for all* i, j.

Proof. Consider an imputation that any player in $\bigcup_{i,j} S_{ij}$ is allocated by $s_{ij}(n) = U_i$ for all (i, j) and n and player c is allocated by $\pi = F - \sum_{i,j} U_i N_{ij} dl_{ij}$. Such an imputation is always feasible. Obviously, any coalition which doesn't consist of player c satisfies coalitional or individual rationality. So does any coalition S which consist of player c because if S consists of n_{ij} players from each subset S_{ij}, its imputation yields $\pi + \sum_{i,j,n} s_{ij}(n) = F - \sum_{i,j} U_i(N_{ij} - n_{ij}) dl_{ij} \geq F - \sum_{i,j}(\partial F/\partial l_{ij}) dl_{ij} - o(dl_{ij}) > F(l_{11} - (N_{11} - n_{11}) dl_{11}, \ldots, l_{LM_L} - (N_{LM_L} - n_{LM_L}) dl_{LM_L}) = v(S)$, which implies that this imputation is located in core. □

Theorem 3.3. *If* $\partial F/\partial l_{ij} \geq U_i$ *for* $(i, j) \in \Upsilon$ *in game* $\eth_{\infty}\left(2^{\Omega}, X_0\right)$*, the following imputation is supported by both nucleolus and Shapley value, and also it is included in core.*

$$E_{ij}(t) = \frac{1}{2}\left(U_i(t) + \frac{\partial F}{\partial l_{ij}}(t)\right) \tag{3.1}$$

[6] Although U has common value for all A, it is allowed to fluctuate over time when dynamics is considered in later sections.

Step 1. Proof of nucleolus. First, we prove that (3.1) is nucleolus starting from the following lemma.

Lemma 3.4. *Consider a coalitional game $\mho_{\mathbf{N}}(\mathcal{S}, X)$ for any $\mathcal{S} \subset 2^{\Omega}$, X and $\mathbf{N}$. Then, if $\partial F/\partial l_{ij} \geq U_i$ for all (i, j), the least core $\Gamma(\varepsilon)$ is characterized by*

$$\varepsilon = -\frac{1}{2}\left(\frac{\partial F}{\partial l_{(i,j)^*}} - U\right) dl_{(i,j)^*} - \frac{1}{2} o\left(dl_{(i,j)^*}\right) \leq 0$$

where $(i, j)^* = \arg\min_{(i,j)} F - F\left(\ldots, l_{ij} - dl_{ij}, \ldots\right)$ and $\partial F/\partial l_{ij}$ is evaluated at $\partial F/\partial l_{ij} = F$.

Proof of Lemma 3.4. Suppose $\left\{s_{ij}(n)\right\} \in \mathcal{S}$ for some $(i, j) \in \Upsilon$, $n \in \Psi_{ij}$. We start by considering a trivial coalition that contains only player $s_{ij}(n)$. Consider a (pre)imputation in an ε-core X for given excess ε. The condition of an ε-core requires $E_{ij}(n)$ to be

$$E_{ij}(n) \geq v\left(s_{ij}(n)\right) - \varepsilon = U_i dl_{ij} - \varepsilon \tag{3.2}$$

for all $n \in M_{ij}$ and $(i, j) \in \Upsilon$. Note that any players in the same subset S_{ij} are symmetric players. Similarly, consider a coalition $\Omega \setminus \left\{s_{ij}(n)\right\}$. Its coalitional payoff is, by total rationality, $F(l_{11}, \ldots, l_{LM_L}) - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - E_{ij}(n)$. The condition of an ε-core requires

$$F - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - E_{ij}(n) \geq v\left(\Omega \setminus \left\{s_{ij}(n)\right\}\right) - \varepsilon = F\left(\ldots, l_{ij} - dl_{ij}, \ldots\right) - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - \varepsilon,$$

where we write the value of $F(l_1, l_2, \ldots, l_M)$ simply as F for concise notation. Then, we obtain

$$\begin{aligned} E_{ij}(n) &\leq F - F\left(\ldots, l_{ij} - dl_{ij}, \ldots\right) + \varepsilon \\ &= \frac{\partial F}{\partial l_{ij}} dl_{ij} + o\left(dl_{ij}\right) + \varepsilon \end{aligned} \tag{3.3}$$

where $o\left(dl_{ij}\right) \geq 0$ by concavity of F. In the space of preimputation $\{X \in \mathbb{R}^{\sum_{i,j} N_{ij}+1}\}$, consider a Cartesian product of an interval on axis $E_{ij}(n)$ which satisfies coalitional rationality for both $\left\{s_{ij}(n)\right\}$ and $\Omega \setminus \left\{s_{ij}(n)\right\}$ for given excess ε, and $\mathbb{R}$ in all other axes. Then, consider an intersection of the above domain with a simplex manifold $\Delta := \{X \in \mathbb{R}^{\sum_{i,j} N_{ij}+1} : J + \sum_{(i,j)\in\Upsilon} \sum_{n\in\Psi_{ij}} E_{ij}(n) = F\left(l_{11}, \ldots, l_{LM_L}\right)\}$. We call it $\mathsf{B}(\left\{s_{ij}(n)\right\}_{n\in\Psi_{ij},(i,j)\in\Upsilon}, \varepsilon)$. From inequalities (3.2) and (3.3),

$$\mathsf{B}\left(\left\{s_{ij}(n)\right\}_{n\in\Psi_{ij},(i,j)\in\Upsilon}, \varepsilon\right) = \begin{cases} \left\{X \in \Delta : U_i dl_{ij} - \varepsilon \leq E_{ij}(n) \leq \dfrac{\partial F}{\partial l_{ij}} dl_{ij} + o\left(dl_{ij}\right) + \varepsilon\right\} \\ \quad \text{if } \varepsilon \geq \dfrac{U - \left[F - F\left(\ldots, l_{ij} - dl_{ij}, \ldots\right)\right]}{2} dl_{ij} \\ \emptyset \quad \text{otherwise} \end{cases} \tag{3.4}$$

Now, consider a coalition S which consists of more than one player $s_{ij}(n)$. Suppose that S consists of $n_{ij} \in \mathbb{N}$ players from each S_{ij} such that $0 \leq n_{ij} \leq N_{ij}$ for all (i, j) and $n_{ij} > 0$ for some (i, j). Then, the payoff of coalition

$\Omega \setminus S$, $F - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - \sum_S E_{ij}(n)$, is

$$F - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - \sum_S E_{ij}(n) \geq F\left(l_{11} - n_{11}dl_{11}, \ldots, l_{LM_L} - n_{LM_L}dl_{LM_L}\right) - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - \varepsilon$$
$$= F - \sum_{(i,j)\in\Upsilon} \frac{\partial F}{\partial l_{ij}} n_{ij}\, dl_{ij} - \sum_{(i,j)\in\Upsilon} o\left(n_{ij}dl_{ij}\right) - \sum_{i,j,n} X_h\left(\left\{s_{ij}(n)\right\}\right) - \varepsilon \tag{3.5}$$

where $o\left(n_{ij}dl_{ij}\right) \geq n_{ij}o\left(dl_{ij}\right) \geq 0$ by concavity of F. The first inequality is strict if there is (i, j) such that $n_{ij} > 1$. On the other hand,

$$\sum_S E_{ij}(n) \geq U_i \sum_{(i,j)\in\Upsilon} n_{ij}\, dl_{ij} - \varepsilon. \tag{3.6}$$

We define domain $\mathsf{B}(S, \varepsilon)$ in the simplex manifold which satisfies coalitional rationality for S and $\Omega\backslash S$ for given excess ε. Denote the set of players' identity numbers for those who belong to both S and Ψ_{ij} by $\Phi_{ij} := \left\{n \in \Psi_{ij} : s_{ij}(n) \in S\right\}$. From equation (3.5) and (3.6), it is given by

$$\mathsf{B}(S, \varepsilon) = \left\{X \in \Delta : \sum_{(i,j)\in\Upsilon} U_i n_{ij}\, dl_{ij} - \varepsilon \leq \sum_{(i,j)\in\Upsilon}\sum_{n\in\Phi_{ij}} E_{ij}(n) \leq \sum_{(i,j)\in\Upsilon} \frac{\partial F}{\partial l_{ij}} n_{ij}\, dl_{ij} + \sum_{(i,j)\in\Upsilon} o\left(n_{ij}dl_{ij}\right) + \varepsilon\right\} \tag{3.7}$$

Note that a symmetric relation

$$\mathsf{B}(S, \varepsilon) = \mathsf{B}(\Omega\backslash S, \varepsilon) \tag{3.8}$$

holds for all $S \in 2^\Omega$ and $\varepsilon \in \mathbb{R}$.

We show that, for any $S \in 2^\Omega$, $\mathsf{B}(S, \varepsilon)$ comprehends the intersection of $\mathsf{B}\left(\left\{s_{ij}(n)\right\}_{n\in\Psi_{ij},(i,j)\in\Upsilon}, \varepsilon\right)$ for all $n \in \Phi_{ij}$ and $(i, j) \in \Upsilon$, namely

$$\bigcap_{n\in\Phi_{ij},(i,j)\in\Upsilon} \mathsf{B}\left(\left\{s_{ij}(n)\right\}, \varepsilon\right) \subseteq \mathsf{B}(S, \varepsilon) \tag{3.9}$$

for any S. By symmetric relation (3.8), it is sufficient to consider S such that $c \notin S$. Pick up a point $y \in \bigcap_{n\in\Phi_{ij},(i,j)\in\Upsilon} \mathsf{B}\left(\left\{s_{ij}(n)\right\}, \varepsilon\right)$. Then, from (3.4), $U_i dl_{ij} - \varepsilon \leq E_{ij}(n) \leq \frac{\partial F}{\partial l_{ij}} dl_{ij} + o\left(dl_{ij}\right) + \varepsilon$ for all $n \in \Phi_{ij}$ and $(i, j) \in \Upsilon$. Summing up this for all (i, j) and n, we obtain $\sum_{(i,j)\in\Upsilon} U_i n_{ij} dl_{ij} - \varepsilon \sum_{(i,j)\in\Upsilon} n_{ij} \leq \sum_{(i,j)\in\Upsilon}\sum_{n\in\Phi_{ij}} E_{ij}(n) \leq \sum_{(i,j)\in\Upsilon} \frac{\partial F}{\partial l_{ij}} n_{ij} dl_{ij} + \sum_{(i,j)\in\Upsilon} n_{ij} o\left(dl_{ij}\right) + \varepsilon \sum_{(i,j)\in\Upsilon} n_{ij}$. Since $\varepsilon \leq 0$ from Lemma 3.2 and from concavity of F, it implies that (3.9) holds and $\mathsf{B}(S, \varepsilon) \neq \emptyset$.

From the above argument, to characterize the ε-core, it is sufficient to focus on coalitional rationality of the family of $\left\{s_{ij}(n)\right\}_{n\in\Phi_{ij},\forall(i,j)\in\Upsilon}$ and $\left\{\Omega\backslash\left\{s_{ij}(n)\right\}\right\}_{n\in\Phi_{ij},\forall(i,j)\in\Upsilon}$. Any conditions given by other coalitions are redundant. It implies that ε-core is characterized by $\bigcap_{n\in\Phi_{ij},\forall(i,j)\in\Upsilon} \mathsf{B}\left(\left\{s_{ij}(n)\right\}, \varepsilon\right)$. Define an interval $I_{ij}(n, \varepsilon) := \left\{E_{ij}(n) : U_i dl_{ij} - \varepsilon \leq E_{ij}(n) \leq \left(\partial F/\partial l_{ij}\right) dl_{ij} + o\left(dl_{ij}\right) + \varepsilon\right\}$. Then, it is equivalent to find an intersection of the interval $\times_{i,j,n} I_{ij}(n, \varepsilon)$ and the simplex Δ. Then, the least core $\Gamma(\varepsilon)$ is obtained by setting ε so that $I_{j^*}(n, \varepsilon)$ degenerates to a point where $(i, j)^*$ is the type (i, j) which has the smallest (discrete) marginal value of production among $(i, j) \in \left\{(i, j) \in \Upsilon : X_h\left(\left\{s_{ij}(n)\right\}\right) = 0\right\}$. Namely, $U_i\, dl_{ij} - \varepsilon = F - F\left(\ldots, l_{(i,j)^*} - dl_{(i,j)^*}, \ldots\right) + \varepsilon$, from which we obtain

$$\varepsilon = -\frac{1}{2}\left[F - F\left(\ldots, l_{(i,j)^*} - dl_{(i,j)^*}, \ldots\right) - U_{i^*} dl_{(i,j)^*}\right] = -\frac{1}{2}\left(\frac{\partial F}{\partial l_{(i,j)^*}} - U_{i^*}\right) dl_{(i,j)^*} - \frac{1}{2} o\left(dl_{(i,j)^*}\right).$$

□

Continuation of Step 1 of Theorem 3.3. All types of workers whose wage rate is such that the present value of employment exceeds that of their marginal productivity will not be employed at all. Therefore, the game is zero-monotone, and the nucleolus coincides with the lexicographic center (Maschler et al. (1979)). By Lemma 3.4, workers of type $(i, j)^*$ whose present value of discrete marginal productivity is the smallest have the maximum excess which equals to ε in $\eth(\mathcal{S}_h, X_h)$. Their payoff $E_{(i,j)^*}(n)$ is

$$E_{(i,j)^*}(n) = v\left(s_{(i,j)^*}(n)\right) - \varepsilon_h = \frac{1}{2}\left(U + \frac{\partial F}{\partial l_{(i,j)^*}}\right) dl_{(i,j)^*} + \frac{1}{2} o\left(dl_{(i,j)^*}\right).$$

By rule (3) of 3.1, the next reduced game $\eth_{\mathbf{N}}(\mathcal{S}_{h+1}, X_{h+1})$ is characterized by $\mathcal{S}_{h+1} = \mathcal{S}_h \setminus \mathcal{T}_h$ where $\mathcal{T}_h = \{S \in \mathcal{S}_h : X_h(S) = v(S) - \varepsilon_h\}$ and $X_{h+1} - X_h = v(S) - \varepsilon_h$ if $S \in \mathcal{T}_h$ and $X_{h+1} - X_h = 0$ otherwise. In this new game, note that (3.4) and (3.7) are independent from X_h. Therefore, exactly the same logic applies to $\eth_{\mathbf{N}}(\mathcal{S}_{h+1}, X_{h+1})$ as in $\eth_{\mathbf{N}}(\mathcal{S}_h, X_h)$, leading to

$$E_{(i,j)^{**}} = v\left(s_{(i,j)^{**}}(n)\right) - \varepsilon_{h+1} = \frac{1}{2}\left(U + \frac{\partial F}{\partial l_{(i,j)^{**}}}\right) dl_{(i,j)^{**}} + \frac{1}{2} o\left(dl_{(i,j)^{**}}\right)$$

where $(i, j)^{**}$ is the type (i, j) which has the smallest discrete marginal productivity among $(i, j) \in \{(i, j) \in \Upsilon : X_{h+1}\left(\left\{s_{ij}(n)\right\}\right) = 0\}$. Repeating the logic above in every reduced games until we get $\eth_{\mathbf{N}}(\mathcal{S}_\infty, X_\infty)$, it will be found in limit that

$$E_{ij}(n) = \frac{1}{2}\left(U + \frac{\partial F}{\partial l_{ij}}\right) dl_{ij} + \frac{1}{2} o\left(dl_{ij}\right)$$

for all (i, j). In $\eth_\infty\left(2^\Omega, X_0\right)$, the last term becomes negligible, obtaining the result.

Step 2. Proof of Shapley value. Choose a player $s_{i'j'}(q)$ for some $(i', j') \in \Upsilon$ and $n' \in \Psi_{i'j'}$ and consider any coalition $\tilde{S}(n_{11}, \ldots, n_{LM_L}; n_c)$ which contains $s_{i'j'}(n')$ where the number of players of type $(i, j) \in \Upsilon$ joining in $\tilde{S}$ by n_{ij} and the number of firm joining in $\tilde{S}$ by n_c (i.e. $n_c = 0$ or 1). Then, $\|\tilde{S}\| = \sum_{i,j} n_{ij} + n_c$ and the "weights" $\gamma(\tilde{S})$ for possible contribution of $s_{i'j'}(n')$ in $\tilde{S}$ is given by

$$\begin{aligned}
\gamma(\tilde{S}) &= \frac{(\sum_{i,j} n_{ij} + n_c - 1)!(\sum_{i,j} \|S_{ij}\| + 1 - \sum_{i,j} n_{ij} - n_c)!}{(\sum_{i,j} \|S_{ij}\| + 1)!} \\
&= B\left(\sum_{i,j} n_{ij} + n_c, \sum_{i,j}\left(\|S_{ij}\| - n_{ij}\right) + (1 - n_c) + 1\right)
\end{aligned}$$

where $B(\cdot, \cdot)$ is a Beta function. Now, suppose that $s_{i',j'}(n')$ is the last player joining $\tilde{S}(n_{11}, \ldots, n_{LM_L}; n_c)$. There are two cases in his contribution to $\tilde{S}(n_{11}, \ldots, n_{LM_L}; n_c)$.

Case 1) $\left(\exists (i^*, j^*) \neq (i, j) \in \Upsilon, \forall n \in \Psi_{i^*j^*}, s_{i^*j^*}(n) \notin \tilde{S}\right) \vee \left(c \notin \tilde{S}\right)$**:** Contribution of player $s_{i'j'}(n')$ is U_i regardless of $\|\tilde{S}\|$.

Case 2) $\left(\forall (i,j)\in\Upsilon,\ \exists n\in\Psi_{ij},\ s_{ij}(n)\in\tilde{S}\right)\wedge\left(c\in\tilde{S}\right)$**:** Contribution of player $s_{i'j'}(n')$ is

$$F\left(n_{11}dl_{11},\ldots,n_{i'j'}dl_{i'j'},\ldots,n_{LM_L}dl_{LM_L}\right)-F\left(n_{11}dl_{11},\ldots,\left(n_{i'j'}-1\right)dl_{i'j'},\ldots,n_{LM_L}dl_{LM_L}\right)$$

where $F(\cdot,\ldots,\cdot)=\left(\left\|\tilde{S}\right\|-n_c\right)U$ if $\exists i,\ n_i=0$.

Therefore, Shapley value of player $s_{i'j'}(n')$, $\phi_{(i',j'),n'}$, defined as a density function in terms of the measure of the type (i',j') player is

$$\begin{aligned}
\phi_{(i'j'),n'}dl_{i'j'} &= \sum_{\{\tilde{S}:n_c=0\}}\gamma\left(\tilde{S}\right)U_i\,dl_{i'j'} \\
&\quad+\sum_{\{\tilde{S}:n_c=1\}}\gamma\left(\tilde{S}\right)\left[F\left(\ldots,n_{i'j'}dl_{i'j'},\ldots\right)-F\left(\ldots,\left(n_{i'j'}-1\right)dl_{i'j'},\ldots\right)\right]dl_{11}\cdots dl_{LM_L} \\
&=\frac{U}{2}\,dl_{i'j'}+\sum_{\{\tilde{S}:n_c=1\}}\gamma\left(\tilde{S}\right)\left[F\left(\ldots,n_{i'j'}dl_{i'j'},\ldots\right)-F\left(\ldots,\left(n_{i'j'}-1\right)dl_{i'j'},\ldots\right)\right]dl_{11}\cdots dl_{LM_L}
\end{aligned}\tag{3.10}$$

where $\sum_{\{\tilde{S}:n_c=1\}}\gamma\left(\tilde{S}\right)=1/2$. Using the relation

$$\gamma\left(n_{11}dl_{11},\ldots,n_{LM_L}dl_{LM_L}\right)=B\left(\sum_{(i,j)\in\Upsilon}\frac{n_{ij}dl_{ij}}{l_{ij}}\left\|S_{ij}\right\|+n_c,\ \sum_{(i,j)\in\Upsilon}\left(1-\frac{n_{ij}dl_{ij}}{l_{ij}}\right)\left\|S_{ij}\right\|+1\right),$$

with a slight abuse of notation,

$$\gamma\left(n_{11}dl_{11},\ldots,n_{LM_L}dl_{LM_L}\right)\quad\to\quad\begin{cases}\infty & \text{if } n_{ij}dl_{ij}=l_{ij}\text{ for all }(i,j)\\ 0 & \text{otherwise}\end{cases}$$

as $dl_{11},\ldots,dl_{LM_L}\to 0$ keeping $\left\|S_{ij}\right\|dl_{ij}=l_{ij}$ for all $(i,j)\in\Upsilon$. It implies that γ is expressed, using generalization of Dirac's δ to multiple dimensions, by

$$\gamma\left(L_{11},\ldots,L_{LM_L}\right)=\delta\left(1-\frac{L_{11}}{l_{11}},\ldots,\ 1-\frac{L_{LM_L}}{l_{LM_L}}\right)$$

where

$$\delta(x_{11},\ldots,x_{LM_L})\quad=\quad\begin{cases}\infty & \text{if } x_{11}=\cdots=x_{LM_L}=0\\ 0 & \text{otherwise}\end{cases}$$

and $\int_0^1\cdots\int_0^1\delta(l_{11},\ldots,l_{LM_L})\,dl_{11}\cdots dl_{LM_L}=1$. Therefore, from (3.10), we obtain

$$\phi_{(i',j'),n'}=\frac{U_{i'}}{2}+\frac{1}{2}\frac{\partial F}{\partial l_{i'j'}}$$

for all $n\in\Psi_{i'j'}$.

Step 3. Proof of core. From Lemma 3.4, $\varepsilon\leq 0$ if $\partial F/\partial l_{ij}\geq U_i$ for the least core of $\mathfrak{D}_{\mathbf{N}}(\Omega)$, which implies that the core includes a nucleolus solution.

□

The above distribution of rent is achieved through wage payment to workers in each period, retaining the rest as profit on the firm side. To obtain a wage function from the above result, we have to obtain an explicit form of E_i and U for any time t, which will be done in Section 4.

4. Wage Function

Taking $z_{ij} := E_{ij} - U_i$ for all $(i, j) \in \Upsilon$ in Bellman equations (2.2) and (2.3), the dimension of the dynamics defined by (2.2) and (2.3) is reduced by one:

$$\dot{\mathbf{z}}_i(t) = A_i(t)\,\mathbf{z}_i(t) - \mathbf{f}_i(t) \tag{4.1}$$

where $\mathbf{z}_i(t) := \begin{pmatrix} z_{i1}(t) \\ z_{i2}(t) \\ \vdots \\ z_{iM_i}(t) \end{pmatrix}$, $A_i(t) := \begin{pmatrix} r(t)+\sigma(t)+g_{i1}\mu_i(t) & g_{i2}\mu_i(t) & \cdots & g_{LM_L}\mu_i(t) \\ g_{i1}\mu_i(t) & r(t)+\sigma(t)+g_{i2}\mu_i(t) & & g_{LM_L}\mu_i(t) \\ \vdots & & \ddots & \vdots \\ g_{i1}\mu_i(t) & g_{i2}\mu_i(t) & \cdots & r(t)+\sigma(t)+g_{LM_L}\mu_i(t) \end{pmatrix}$

and $\mathbf{f}(t) := \begin{pmatrix} w_{11}(t) - b_1(t) \\ w_{12}(t) - b_1(t) \\ \vdots \\ w_{LM_L}(t) - b_L(t) \end{pmatrix}$. Note that $A(t)$ has eigenvalues $r(t) + \sigma(t)$ with multiplicity $(M - 1)$ and $r(t) + \sigma(t) + \mu(t)$ with multiplicity one.[7] It can be confirmed that the following provides the elementary matrix $\Phi(t, s)$:

$$\Phi_i(t,s) := e^{\int_s^t A_i(q)dq} = e^{\int_s^t \beta(q)\,dq}\left[I + \left(e^{\int_s^t \mu_i(q)\,dq} - 1\right)G\right]$$

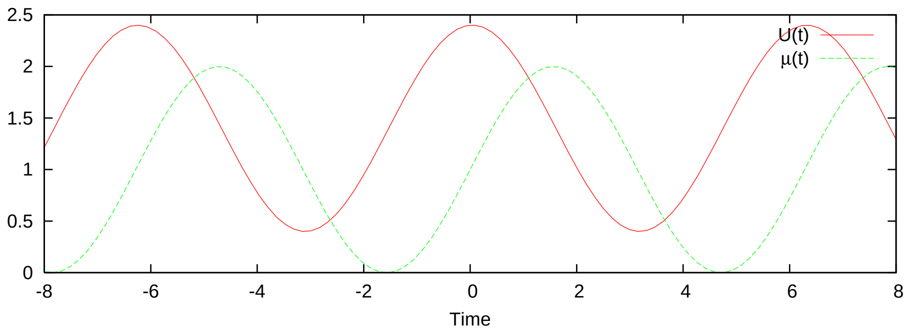
where I is an identity matrix, $\beta(q) := r(q) + \sigma(q)$ and G_i is an "expectation matrix"

$$G_i = \begin{pmatrix} g_{i1} & g_{i2} & \cdots & g_{iM_i} \\ \vdots & \vdots & & \vdots \\ g_{i1} & g_{i2} & \cdots & g_{iM_i} \end{pmatrix}.$$

Namely, $\mathbf{z}_i(t) = \Phi_i(t, s)\,\mathbf{c}$ for any $\mathbf{c} \in \mathbb{R}^2$ solves the accompanying homogeneous equation to (4.1). Then, the solution to (4.1) is given by $\mathbf{z}(t) = \Phi(t, 0)[\mathbf{z}_0 - \int_0^t \Phi(s, 0)^{-1}\,\mathbf{f}(s)\,ds] = e^{\int_0^t A(q)dq}[\mathbf{z}_0 - \int_0^t e^{-\int_0^s A(q)dq}\,\mathbf{f}(s)\,ds]$ for any initial value $\mathbf{z}(0) = \mathbf{z}_0 = (z_{10}, z_{20})$. Note that $\Phi(t, s)^{-1} = e^{-\int_s^t A(q)dq}$. For the no-Ponzi game condition to hold, the initial value must be set at $\mathbf{z}_0 = \int_0^\infty \Phi(s, 0)^{-1}\,\mathbf{f}(s)\,ds$ in which integration is bounded. For such an initial value, $\mathbf{z}(t) = \int_t^\infty \Phi(s, t)^{-1}\,\mathbf{f}(s)\,ds = \int_t^\infty e^{-\int_t^s A(q)dq}\,\mathbf{f}(s)\,ds$. Using the fact that $[I + (\alpha - 1)G]^{-1} = I + (\alpha^{-1} - 1)G$ for any scholar α, it is found that

$$\Phi_i\,(s,t)^{-1} = e^{-\int_t^s \beta(q)\,dq}\left[I + \left(e^{\int_t^s -\mu_i(q)\,dq} - 1\right)G_i\right].$$

[7]To obtain this simple result, it is critical to assume that separation rate is common for all worker types. We still continue to use the notation σ_{ij} in other places for future extension, however, it should be understood that $\sigma_{ij} = \sigma$ for all i, j.



U anticipates the fluctuation of μ with time lag.

FIGURE 4.1. Response of $U(t)$

Namely,

$$z_{ij}(t) = \int_t^\infty e^{-\int_t^s (r+\sigma)} \left[\left(w_{ij} - b_i\right) - \mathsf{E}_j \left(w_{ij} - b_i\right) + e^{-\int_t^s \mu_i} \mathsf{E}_j \left(w_j - b_i\right) \right] \tag{4.2}$$

where expectation E is taken over all possible worker types. Solving differential equation (2.2) for U_i using (4.2),

$$U_i(t) = \int_t^\infty e^{-\int_t^s r(q)dq} \left[b_i(s) + \mu_i(s) \int_s^\infty \mathsf{E}_j \left(w_{ij}(\xi) - b_i(\xi) \right) e^{-\int_s^\xi \alpha_i(q)dq} d\xi \right] ds \tag{4.3}$$

Similarly, we obtain the value function of employment for each type.

$$\begin{aligned} E_{ij}(t) = &\int_t^\infty e^{-\int_t^s r(q)\,dq} w_{ij}(s) ds \\ &+ \int_t^\infty ds \int_s^\infty e^{-\int_t^\xi r(q)\,dq} \sigma(s) e^{-\int_s^\xi \sigma(q)dq} \left[-\left(w_{ij} - b_i\right) + \left(1 - e^{-\int_s^\xi \mu_i(q)dq}\right) \mathsf{E}_j \left(w_{ij}(\xi) - b_i(\xi) \right) \right] d\xi \end{aligned} \tag{4.4}$$

The unemployment value function is the sum of the discounted series of unemployment benefit and the expected discounted series of capital gain arising from matching. Note that the value of unemployment responds to the change of matching probability with time lag. Suppose that the capital gain of matching, unemployment benefit and interest rate are fixed over time and only matching probability periodically fluctuates, say $\mu(t) = (1 + \sin t)/2$. Then, the value of unemployment becomes $U(t) = a + b(r \sin t + \cos t)$ where a and b are fixed coefficients (see Figure 4.1). It implies the bargaining power of workers fluctuates with time lag *before* the change of matching probability under rational forecast, which may make the adjustment via wage rate in the labor market imperfect. This is not mere a special case when intertemporal fluctuation of μ has a sine curve. If μ has a general functional form which is absolute integrable in terms of time, it is shown that the same property holds by Fourier transformation.

Proposition 4.1. *Wage rate at time t is given by*

$$\begin{aligned} w_{ij}(t) = \mathsf{E}_h \mathfrak{F}_{ih}(t) - \frac{1}{2}\left(\mathfrak{F}_{ij}(t) - b_i(t)\right) - \frac{1}{2}\sigma(t) \int_t^\infty \left(\mathsf{E}_h \mathfrak{F}_{ih}(\xi) - \mathfrak{F}_{ij}(\xi)\right) e^{-\int_t^\xi r(q)dq} d\xi \\ + \frac{1}{2}\left(\frac{\mu(t)}{2} - \sigma(t)\right) \int_t^\infty \left(\mathsf{E}_h \mathfrak{F}_{ih}(\xi) - b_i(\xi)\right) e^{-\int_t^\xi (r(q)+3\mu_i/2)dq} d\xi \end{aligned}$$

where $\mathfrak{F}_i$ is capital gain of marginal value of production, i.e. $\mathfrak{F}_{ij} := r\partial F/\partial l_{ij} - \partial^2 F/\partial t \partial l_{ij}$.

Proof. Theorem 3.3 implies $\partial^2 F/\partial t\partial l_i = 2\dot{E}_i - \dot{U}$. Applying (2.2), (2.3) and (3.1),

$$\frac{\partial^2 F(t)}{\partial t \partial l_{ij}} = r(t)\frac{\partial F(t)}{\partial l_{ij}} - \Big(2w_{ij}(t) - b_i(t)\Big) + 2\sigma(t) z_{ij}(t) - \mu_i(t) \mathsf{E}_h\left(w_{ih} - b_i\right) \tag{4.5}$$

Taking difference of (4.5) for any i and $j \neq i$, we obtain a Volterra integral equation of the second kind concerning w_1 and w_2.[8]

$$\Big(w_{ij}(t) - w_{ih}(t)\Big) - \sigma_i(t)\int_t^\infty \Big(w_{ij}(\xi) - w_{ih}(\xi)\Big) e^{-\int_t^\xi (\sigma_i + r)} d\xi = \frac{1}{2}\left[r(t)\left(\frac{\partial F(t)}{\partial l_{ij}} - \frac{\partial F(t)}{\partial l_{ih}}\right) - \left(\frac{\partial^2 F(t)}{\partial t \partial l_{ij}} - \frac{\partial^2 F(t)}{\partial t \partial l_{ih}}\right)\right] \tag{4.6}$$

On the other hand, taking expectation of (4.5) yields

$$\mathsf{E}_h\left(w_{ih}(t) - b_i(t)\right) - \left(\sigma_i(t) - \frac{\mu_i(t)}{2}\right)\int_t^\infty \mathsf{E}_{ih}\left(w_{ih}(\xi) - b_i(\xi)\right) e^{-\int_t^\xi (r+\sigma_i+\mu_i)} d\xi = \frac{1}{2}\mathsf{E}_h\left(r(t)\frac{\partial F(t)}{\partial l_{ih}} - \frac{\partial^2 F(t)}{\partial t \partial l_{ih}} - b_i(t)\right) \tag{4.7}$$

The above operations suggest that it is beneficial to define new variables $Y_{ij}(t)$ ($j = 1, 2, \ldots, M_i$) as follows.

$$\begin{pmatrix} Y_{i1}(t) \\ Y_{i2}(t) \\ Y_{i3}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix} := \left(\begin{array}{c|cccc} g_{i1} & g_{i2} & g_{i3} & \cdots & g_{iM_i} \\ \hline 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{array}\right) \begin{pmatrix} w_{i1}(t) - b_i(t) \\ w_{i2}(t) - b_i(t) \\ w_{i3}(t) - b_i(t) \\ \vdots \\ w_{iM_i}(t) - b_i(t) \end{pmatrix}$$

Observe that the above conversion matrix is the same as the eigenvector matrix of $A_i(t)$. By this change of variables, we can "diagonalize" the simultaneous integral equations concerning w_1 and w_2 (4.6) and (4.7). Namely,

$$\begin{pmatrix} Y_{i1}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix} - \int_t^\infty \begin{pmatrix} K^i_{11}(t,\xi) & & O \\ & \ddots & \\ O & & K^i_{MM}(t,\xi) \end{pmatrix} \begin{pmatrix} Y_{i1}(\xi) \\ \vdots \\ Y_{iM_i}(\xi) \end{pmatrix} d\xi = \frac{1}{2}\begin{pmatrix} h_{i1} \\ \vdots \\ h_{iM} \end{pmatrix}$$

where

$$\begin{aligned} K^i_{11}(t,\xi) &:= \left(\sigma_i(t) - \frac{\mu_i(t)}{2}\right) e^{-\int_t^\xi (r+\sigma_i+\mu_i)} \\ K^i_{jj}(t,\xi) &:= \sigma_i(t)\, e^{-\int_t^\xi (r+\sigma_i)} \qquad \text{(for all } j = 2, \ldots, M) \\ h_{i1}(t) &:= \mathsf{E}_h\left(r(t)\frac{\partial F(t)}{\partial l_{ih}} - \frac{\partial^2 F(t)}{\partial t \partial l_{ih}} - b_i(t)\right) \\ h_{ij}(t) &:= r(t)\left(\frac{\partial F(t)}{\partial l_{ij}} - \frac{\partial F(t)}{\partial l_{1j}}\right) - \left(\frac{\partial^2 F(t)}{\partial t \partial l_{ij}} - \frac{\partial^2 F(t)}{\partial t \partial l_{1j}}\right) \qquad \text{(for all } j = 2, \ldots, M) \end{aligned}$$

---

[8]Note that it is impossible to obtain a differential equation by taking time derivative of this equation since t resides inside of the integration. It is a general consequence of non-stationarity.

and the integration is applied element-wise. Then, the solution to this equation is given by

$$\begin{pmatrix} Y_{i1}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} h_{i1}(t) \\ \vdots \\ h_{iM_i}(t) \end{pmatrix} - \frac{1}{2}\int_t^\infty \begin{pmatrix} G^i_{11}(t,\xi) & & O \\ & \ddots & \\ O & & G^i_{MM}(t,\xi) \end{pmatrix}\begin{pmatrix} h_{i1}(\xi) \\ \vdots \\ h_{iM_i}(\xi) \end{pmatrix} d\xi$$

where $G^i_{jj}(t,\xi) := -\sum_{h=1}^\infty \overset{*}{K}{}^{i\,h}_{jj}(t,\xi)$ for $j = 1,2,\ldots,M_i$. Iterated kernel $\overset{*}{K}{}^n$ is defined by $\overset{*}{K}{}^n := \underbrace{K * K * \cdots * K}_{n}$ and $K * L$ denotes the composition of the first kind defined as $K(t,\xi) * L(t,\xi) = \int_t^\xi K(t,\tau)\,L(\tau,\xi)\,d\tau$ (see Yokota (2006) for example). Since

$$\begin{aligned}
(\overset{*}{K}{}^i_{11})^n(t,\xi) &= \left(\sigma_i(t) - \frac{\mu_i}{2}(t)\right) e^{-\int_t^\xi (r+\sigma_i+\mu_i)} \frac{\left[\int_t^\xi \left(\sigma_i(s) - \frac{\mu_i(s)}{2}\right) ds\right]^{n-1}}{(n-1)!} \\
(\overset{*}{K}{}^i_{jj})^n(t,\xi) &= \sigma_i(t) e^{-\int_t^\xi (r+\sigma_i)} \frac{\left[\int_t^\xi \sigma_i(s)\,ds\right]^{n-1}}{(n-1)!} \qquad \text{(for all } j = 2,\ldots,M),
\end{aligned}$$

we obtain

$$\begin{aligned}
G^i_{11}(t,\xi) &= \left(\sigma_i(t) - \frac{\mu_i(t)}{2}\right) e^{-\int_t^\xi (r+2\sigma_i+\mu_i/2)} \\
G^i_{jj}(t,\xi) &= -\sigma_i(t)\, e^{-\int_t^\xi (r+2\sigma_i)} \qquad \text{(for all } j = 2,\ldots,M)
\end{aligned}$$

and the solution for $Y_{ij}(t)$.

$$\begin{aligned}
Y_{i1}(t) &= \frac{1}{2}h_{i1}(t) + \frac{1}{2}\left(\sigma_i(t) - \frac{\mu_i(t)}{2}\right)\int_t^\infty e^{-\int_t^\xi (r+3\mu_i/2)} h_{i1}(\xi)\,d\xi \\
Y_{ij}(t) &= \frac{1}{2}h_{ij}(t) - \frac{1}{2}\sigma_i(t)\int_t^\infty e^{-\int_t^\xi r} h_{ij}(\xi)\,d\xi \qquad \text{(for all } j = 2,\ldots,M)
\end{aligned}$$

Inverting back to $w_{ij}(t)$ using

$$\begin{pmatrix} w_{i1}(t) - b_i(t) \\ w_{i2}(t) - b_i(t) \\ \vdots \\ w_{iM}(t) - b_i(t) \end{pmatrix} = \left[\begin{pmatrix} 1 & g_{i2} & \cdots & g_{iM_i} \\ 1 & g_{i2} & \cdots & g_{iM_i} \\ \vdots & \vdots & & \vdots \\ 1 & g_{i2} & \cdots & g_{iM_i} \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & O \\ \vdots & & \ddots & \\ 0 & O & & 1 \end{pmatrix}\right]\begin{pmatrix} Y_{i1}(t) \\ Y_{i2}(t) \\ \vdots \\ Y_{iM_i}(t) \end{pmatrix},$$

the result of the proposition is derived. □

Now, we can present some of the properties about wages. Generally, the expected present value of wages is greater than that of unemployment benefits as far as the marginal contribution to the value of production is greater than the value of unemployment. In addition, if the marginal contribution to the value of production is decreasing over time, wage rate is greater than unemployment benefit.

Proposition 4.2. *If $\partial F/\partial l_{ij} > U_i$ for all (i, j), then $\int_t^\infty \mathsf{E}_j w_{ij}(s) e^{-\int \alpha_i(q)dq} ds > \int_t^\infty b_i(s)\, e^{-\int \alpha_i(q)dq} ds$.*

Proof. From Theorem 3.3, the condition $\partial F/\partial l_{ij} > U_i$ implies $z_{ij} = (\partial F/\partial l_{ij} - U_i)/2 > 0$. Namely,

$$z_{ij} = \int_t^\infty \mathsf{E}_h[w_{ih} - b_i]\, e^{-\int \alpha_i(q)\,dq} ds - \int_t^\infty \left\{\mathsf{E}_h[w_{ih} - b_i] - \left(w_{ij} - b_i\right)\right\} e^{-\int \beta(q)\,dq} ds > 0$$

must hold for all (i, j) from (4.2), which yields

$$\int_t^\infty \mathsf{E}_h[w_{ih} - b_i]\, e^{-\int \alpha_i(q)\,dq} ds > \max_j \int_t^\infty \left\{\mathsf{E}_h[w_{ih} - b_i] - \left(w_{ij} - b_i\right)\right\} e^{-\int \beta(q)\,dq} ds \geq 0$$

to obtain the result. □

Proposition 4.3. *If $\partial^2 F/\partial t \partial l_{ij} - \dot{U}_i \leq 0$, then $w_{ij}(t) > b_i(t)$ for all $(i, j) \in \Upsilon$ and t.*

Proof. From Theorem 3.3,

$$\dot{E}_{ij}(t) = \frac{1}{2}\left(\dot{U}_i(t) + \frac{\partial^2 F}{\partial t \partial l_{ij}}(t)\right) \tag{4.8}$$

which yields

$$\dot{E}_{ij}(t) - \dot{U}_i(t) = \frac{1}{2}\left(\frac{\partial^2 F}{\partial t \partial l_{ij}}(t) - \dot{U}_i(t)\right) \leq 0$$

From (2.2) and (2.3)

$$w_{ij} - b_i = (r + \sigma_i)\left(E_{ij} - U_i\right) + \mu \mathsf{E}_h[E_{ih} - U_i] - \left(\dot{E}_{ij} - \dot{U}_i\right) > 0.$$

□

The condition of Proposition 4.3 obviously holds at a steady state either when the demand constraint is binding or unbinding. On the other hand, when b is expected to rise only for a sufficiently short period of time from now on, it can happen that wage rate becomes temporarily smaller than unemployment benefit whereas $E_i > U$ still holds and thus workers do not willing to quit the current jobs.

5. Production Plan

The result of the previous section shows that wage is a function of employment. Knowing the wage schedule, a firm determines optimal policy on vacancy post and dismissal. The optimal problem for a firm is given by

$$J(\mathbf{l}, y) = \max_{m,x} \int_t^\infty \left[f(\mathbf{l}) - \mathbf{w}(\mathbf{l}) \cdot \mathbf{l} - \sum_{i=1}^L \kappa_i(m_i)\right] \exp\left[-\int_t^\xi r(\tau) d\tau\right] d\xi \tag{P}$$

subject to

$$\dot{l}_{ij} = g_{ij}\psi(\theta_i)\, m_i - \sigma_{ij} l_{ij} - x_{ij}, \quad \forall i = 1, \ldots, L;\ j = 1, \ldots, M_i \tag{5.1}$$

$$0 \leq x_{ij} \leq X \qquad \forall i, j \tag{5.2}$$

$$m_i \geq 0 \qquad \forall i \tag{5.3}$$

$$f(\mathbf{l}) \leq y \tag{5.4}$$

$$l_{ij} \geq 0 \qquad \forall i, j \tag{5.5}$$

$$l_{ij}(0), \forall i, j \text{ given.}$$

where $\mathbf{l} \in \mathbb{R}_+^{\sum_i^L M_i}$ is a vector of labor such that $\mathbf{l} := (\mathbf{l}_1, \ldots, \mathbf{l}_L)$ and $\mathbf{l}_i := (l_{i1}, \ldots, l_{iM_i})$, parameters y, g, θ, σ are generally time-dependent and X is an arbitrarily large number. We assume that X is large enough so that a firm can accommodate any negative change of y. Also, it is assumed that r is bounded and $r(t) \nrightarrow 0$ as $t \to \infty$ and y is differentiable up to second degree concerning to time as far as it is expected. Note that it will be proven later that labor market is always in the state of excess demand. Walras Law implies that the goods market is always in the state of excess supply *regardless the relative price between output goods and labor*. On the other hand, the presence of a convex vacancy cost function prohibits discrete increase of employment, which implies that aggregate production and income can grow only continuously from the current level, and thus the excess supply in the goods market will not be resolved. In general, y should be interpreted as potential demand which is the level of current demand that economic agents believe to exist and it becomes effective demand once the constraint becomes binding *in equilibrium*. The state that potential demand strictly exceeds effective demand is the state where there is coordination of expectation among economic agents. They somehow believe ——it may be because there is some intertemporally binding factor on the demand side or may be simple enthusiasm—— that the current economy can accept production up to the size of y. If such "potential" part of demand is already exhausted, differential change of demand from the current state totally depends on the coordinated differential change of expectation among economic agents. Therefore, the demand constraint (5.4) is required. Removing this constraint can lead the model to inconsistency.

The assumption that production capacity cannot exceed the demand constraint *at any moment* is obviously too strict. If downward shock of demand is expected to be only temporary, it would be optimal for a firm to keep current labor force if hiring cost in future exceeds wage payment to redundant labor force for the period of temporary recession. However, incorporating this possibility introduces indifferentiability in the performance index,[9] which would significantly complicate the analysis. Therefore, knowing that the model presented here retains characteristics that it tends to respond too sharply to high-frequency business cycles downward than the more "realistic" model, we are going to adopt the former to make extensive analysis feasible.

Suppose that the density function of potential demand in the market is given by $n_t(\cdot)$ at time t. Then, the aggregate potential demand Y is given by $Y = \int_0^\infty n(y)\,dy$. In general, Y changes intertemporally, however there is no definitive rule how change of Y alters functional form of $n(\cdot)$. It seems that the present model allows a room for product differentiation and various sales strategies in a competitive environment. If there is a potential entrepreneur whose technology ——including technology of production, design, marketing, etc.—— is so attractive that he gets

[9] The model should be formulated in such a way that the instantaneous profit function is $\min\{y, f(l_1, l_2)\} - \sum w_i l_i - \kappa(m)$ and the state constraint (5.4) is removed.

potential demand $y \gg 0$, on the day he appears in the market he immediately attracts that size of demand. He need not start the business with potential demand in neighborhood of zero. Therefore, the no-entrant condition should be given by $J(0,0,y) = 0$ for all $y \in \mathbb{R}_+$.

Denote the costate variables corresponding to each transition equation of l_{ij} by $\tilde{\lambda}_{ij}$. An augmented Hamiltonian H is defined by

$$H(\xi) := f(\mathbf{l}) - \mathbf{w}(\mathbf{l}) \cdot \mathbf{l} - \sum_{i=1}^{L} \kappa_i(m_i) + \sum_{i,j} \lambda_{ij} \left(\phi_{ij} m_i - \sigma_{ij} l_{ij} - x_{ij}\right) + \mu_0 \left(\dot{y} - \sum_{i=ij} \frac{\partial f}{\partial l_{ij}} \dot{l}_{ij}\right) + \sum_{i,j} \mu_{ij}^1 x_{ij} + \sum_{i,j} \mu_{ij}^2 \left(X - x_{ij}\right) + \sum_i \gamma_i m_i \tag{5.6}$$

where $R(t,\xi) := \int_t^{\xi} r(\tau)\, d\tau$ and μ_0, $\mu_{ij}^n \geq 0$ for $\forall i, j, n$ and $\gamma_i \geq 0$ for $\forall i$ are Lagrange multipliers such that any terms including them are zero. From maximization of Hamiltonian function, optimal conditions for m_i are given by

$$\kappa_i'(m_i) = \sum_j \phi_{ij}\left(\lambda_{ij} - \mu_0 f_{ij}\right) + \gamma_i \tag{5.7}$$

$$\gamma_i m_i = 0 \tag{5.8}$$

$$\lambda_{ij} - \mu_0 f_{ij} = \mu_{ij}^1 - \mu_{ij}^2 \tag{5.9}$$

where $f_{ij} := \partial f / \partial l_{ij}$, and costate dynamics is given by

$$\dot{\lambda}_{ij} = \left(r + \sigma_{ij}\right)\lambda_{ij} + \mu_0\left(\dot{f}_{ij} - \sigma_{ij} f_{ij}\right) - \left(f_{ij} - c_{ij}\right) \qquad \forall i, j \tag{5.10}$$

where $\dot{f}_{ij} := \sum_{a,b} (\partial^2 f / \partial l_{ij} \partial l_{ab})\, \dot{l}_{ab}$, $\mu_0 > 0$ when the demand constraint is binding and $\mu_0 = 0$ when not.

5.1. **Optimal control.**

(a) Unsaturated demand case. If the demand condition (5.4) is not binding, we have $\mu_0 = 0$. Then, the optimal condition for x is given by

$$x_{ij} = \begin{cases} 0 & \text{if } \lambda_{ij} > 0 \\ X & \text{if } \lambda_{ij} < 0 \end{cases} \qquad \forall i, j \tag{5.11}$$

Proposition 5.1. *When $f(\mathbf{l}) < y$, if $\sum_j \phi_{ij}\lambda_{ij} > 0$, then $m_i > 0$. If $\sum_j \phi_{ij}\lambda_{ij} \leq 0$, then $m_i = 0$.*

Proof. If $\sum_j \phi_i \lambda_i > 0$, the right-hand side of equation (5.7) is strictly positive, which implies $m_i > 0$. If $\sum_j \phi_{ij}\lambda_{ij} < 0$, then $\gamma_i > 0$ since the left-hand side of equation (5.7) must be non-negative. From equation (5.8), it implies $m_i = 0$. If $\sum_j \phi_{ij}\lambda_{ij} = 0$, then equation (5.7) becomes $\kappa'(m_i)\, e^{-R(t,\xi)} = \gamma_i$. If we assume $\gamma_i > 0$, equation (5.7) implies $m_i > 0$, contradicting equation (5.8). Thus, $\gamma_i = m_i = 0$. □

Corollary 5.2. *When $f(\mathbf{l}) < y$, if $x_{ij} > 0$ for all j, then $m = 0$. Equivalently, if $m_i > 0$, then $x_{ij} = 0$ for all j.*

It shows that if all types of workers are abundant, there will be no vacancy posting. Optimal combination of m_i and x_i is characterized by domains in the space of λ's separated by $(M_i + 1)$ planes: $\sum_j \phi_{ij}\lambda_{ij} = 0$ and $\lambda_{ij} = 0$ for j, as shown in Figure 5.1 for $M_i = 2$ case.

(b) Saturated demand case. When the effective demand condition (5.4) is binding, the Hamiltonian function is maximized under the constraint on controls $\sum_{i,j} f_{ij}\dot{l}_{ij} = \dot{y}$ or

$$\sum_i \left(\sum_j \phi_{ij} f_{ij} \right) m_i = \dot{y} + \sum_{i,j} \left(\sigma_{ij} l_{ij} + x_{ij} \right) \tag{5.12}$$

with other binding constraints. Of course, the effective demand constraint also must hold as far as the path is on the surface of the constraint, however it is redundant except a point of the path which provides an initial condition.

Proposition 5.3. *Define* $k^i_{ab}(\lambda_a, \lambda_b; \mathbf{l}) := \lambda_{ia}/f_{ia} - \lambda_{ib}/f_{ib}$.

(1) *If* $\sum_a \phi_{ia} f_{ia} k^i_{aj} \le \kappa'_i(\bar{m}_i)$ *for all i and j where* $\bar{m}_i$ *is a solution to*

$$\dot{y} = \sum_i \left(\sum_j \phi_{ij} f_{ij} \right) \bar{m}_i - \sum_i \sum_j \sigma_{ij} f_{ij} l_{ij}$$
$$\frac{\sum_a \phi_{ia}\lambda_{ia} - \kappa'_i(\bar{m}_i)}{\sum_a \phi_{ia} f_{ia}} = \frac{\sum_a \phi_{i'a}\lambda_{i'a} - \kappa'_{i'}(\bar{m}_{i'})}{\sum_a \phi_{i'a} f_{i'a}}, \quad \forall i, i',$$

then $m^*_i = \bar{m}_i$ *and* $x_{ij} = 0$ *for all i and j.*

(2) *If set* $S := \{(i, j) : \sum_a \phi_{ia} f_{ia} k^i_{aj} > \kappa'_i(\bar{m}_i)\}$ *is non-empty, then* m_i *is determined by*

$$\kappa'_{i'}(m_{i'}) = \sum_a \phi_{i'a} f_{i'a} \left(\frac{\lambda_{i'a}}{f_{i'a}} - \frac{\lambda_{i'j}}{f_{i'j}} \right) > \kappa'_{i'}(\bar{m}_{i'}) \qquad \forall i' \in S$$
$$\frac{\sum_a \phi_{ia}\lambda_{ia} - \kappa'_i(\bar{m}_i)}{\sum_a \phi_{ia} f_{ia}} = \frac{\sum_a \phi_{i'a}\lambda_{i'a} - \kappa'_{i'}(\bar{m}_{i'})}{\sum_a \phi_{i'a} f_{i'a}}, \qquad \forall i \notin S.$$

On the other hand, $x_{ij} = 0$ *for all* $(i, j) \notin S$ *and* $x_{i'j'}$ *for all* $(i', j') \in S$ *is given by*

$$\sum_{(i',j') \in S} f_{i'j'} x_{i'j'} = \sum_i \left(\sum_j \phi_{ij} f_{ij} \right) \bar{m}_i - \sum_i \sum_j \sigma_{ij} f_{ij} l_{ij} - \dot{y}$$

and distribution among $x_{i'j'}$*'s is indeterminate.*

Proof. Define $A_{ij} := \lambda_{ij} - \mu_0 f_{ij}$. From (5.9),

$$x_{ij} = \begin{cases} 0 & \text{if } A_{ij} > 0 \\ [0, X] & \text{if } A_{ij} = 0 \\ X & \text{if } A_{ij} < 0 \end{cases}$$

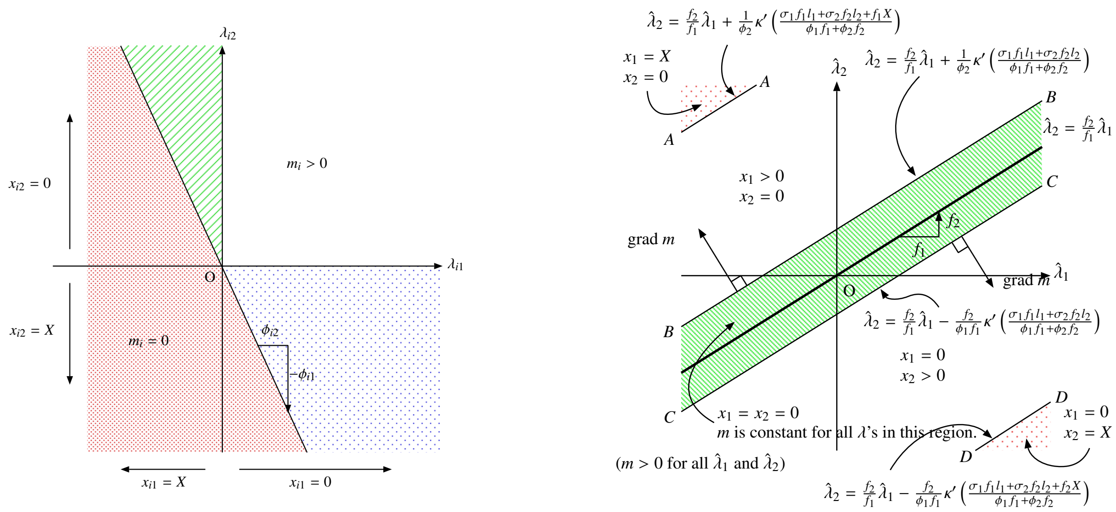


FIGURE 5.1. Optimal control

It is assumed that X is extremely large and the condition $x \leq X$ is never binding, which implies that $A_{ij} \geq 0$ for all i, j and thus $\sum_a \phi_{ia} A_{ia} \geq 0$ for all i. Then, since $\gamma_i = 0$ for all i,

$$\kappa_i'(m_i) = \sum_a \phi_{ia} A_{ia}.$$

Solving this obtains

$$\mu_0 = \frac{\sum_a \phi_{ia}\lambda_{ia} - \kappa_i'(m_i)}{\sum_a \phi_{ia} f_{ia}} \tag{5.13}$$

for all i.

First, suppose $A_{ij} > 0$ for all i and j. Then, $x_{ij} = 0$ for all i and j. (5.12) and (5.13) together with $x_{ij} = 0$ determines m_i which is common for all range in $A_{ij} > 0$. Let us denote it by $\bar{m}_i$. Then, the range $A_{ij} > 0$, $\forall i, j$ is equivalent to $\sum_a \phi_{ia} f_{ia} k_{aj}^i < \kappa_i'(\bar{m}_i)$ for all i and j. Next, suppose that there exist some i' and j' such that $A_{i'j'} = 0$. Then, from $\mu_0 = \lambda_{i'j'}/f_{i'j'}$,

$$\kappa_{i'}'(m_{i'}) = \sum_a \phi_{i'a} f_{i'a}\left(\frac{\lambda_{i'a}}{f_{i'a}} - \frac{\lambda_{i'j'}}{f_{i'j'}}\right)$$

From the demand constraint (5.12), $\sum_i \left(\sum_a \phi_{ia} f_{ia}\right)(m_i - \bar{m}_i) > 0$. On the other hand, from (5.13), if $m_i \gtreqless \bar{m}_i$ for some i, then $m_j \gtreqless \bar{m}_j$ for any j. These leads to $m_i > \bar{m}_i$ for all i. □

Optimal control for each λ for the case of two undeclarable types is shown in Figure 5.1. $|k_{ab}^i|$ can be interpreted as pressure that represents the necessity of structural change in employment composition between type a and b. If the pressure is relatively weak, the structural change is achieved solely through the adjustment of new employment and natural separation. As the pressure grows, the firm is compelled to adopt dismissal. The bandwidth of the domain $x_1 = x_2 = 0$ positively depends on l's.

Note that the analysis made here does not fully characterize a more general case in which more than three kinds of labor are used for production. Adjustment of employment composition without dismissal will be used in the two-labor case to move toward the direction of a bounded steady state, which is reasonable because redundant

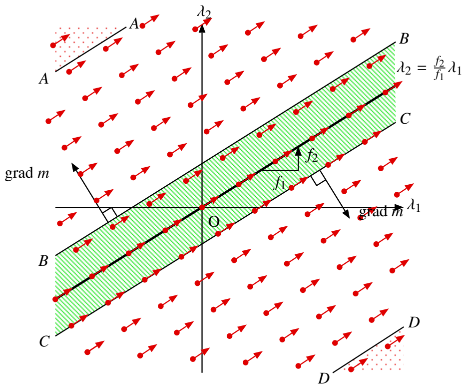


FIGURE 5.2. Vector field in the conjugate space brought by the demand constraint in the configuration space such that its Hamiltonian is the first integral in the original Hamiltonian field

job posting is costly and the dynamics on the surface is one-dimensional. However, when the surface manifold has more than two dimension, the direction toward a bounded steady state can be different from the one toward a long-run optimal state. The conjecture in such a general case is that there would appear a turnpike property in dynamics.

The linear structure of the optimal control on the demand constraint shown in Figure 5.1 is a direct consequence of the presence of the demand constraint. Note that all borders that separate the domains of the optimal control in Figure 5.1 are drawn as parallel lines with the same slope f_2/f_1. This is not mere a coincidence. Consider a transformation of variables along these borders. Namely, introduce new variables $(\Lambda_{11}, \ldots, \Lambda_{MN})$ defined by

$$\begin{pmatrix} \Lambda_{11} \\ \vdots \\ \Lambda_{MN} \end{pmatrix} = \begin{pmatrix} \lambda_{11} \\ \vdots \\ \lambda_{MN} \end{pmatrix} + \begin{pmatrix} f_{11} \\ \vdots \\ f_{MN} \end{pmatrix} s =: \varphi_s \tag{5.14}$$

where s is an arbitrary parameter. Obviously, the optimal control is invariant under this transformation φ_s. Moreover, it is easily confirmed that φ_s also leaves the value of Hamiltonian invariant, which means that this transformation is a *canonical transformation*. Now, consider a vector field $\mathbf{X}_G$ attached to equation (5.14). Then, φ_s is a one-parameter group of transformation with parameter s which vector field $\mathbf{X}_G$ generates. The vector field $\mathbf{X}_G$ can be obtained by the infinitesimal transformation of this group which is

$$\left.\frac{d\varphi_s}{ds}\right|_{s=0} = \sum_{i,j} f_{ij} \frac{\partial}{\partial \lambda_{ij}}$$

where $\partial/\partial\lambda_{ij}$ is the basis of the tangent space. Then, we find out that Hamiltonian G of the vector field $\mathbf{X}_G$ is actually $G = f(\mathbf{l}) - y$. By Noether's Theorem, G satisfies the relation $\{G, H\} = 0$ where $\{\cdot\}$ is the Poisson's braces, which implies that G is a first integral of the Hamilton dynamics given by H.

5.2. **Costate dynamics out of bounded surface.** λ is an *influence function* which shows the impact of the marginal change of the initial state value l on the present value of profits J. It equals the present value of a sequence of marginal profit of labor in which discount rate is the sum of interest rate and separation rate when the demand constraint is unbinding.

Proposition 5.4. *Let $\bar{t}^e \in T^e$ is the first entering time after t. Costate variables when the state constraint is not binding is given by*

$$\lambda_{ij}(t) = \int_t^{\bar{t}^e} \left(\frac{\partial f}{\partial l_{ij}} - \sum_{j=1}^{L} \frac{\partial c_{ij}}{\partial l_{ij}} \right) e^{-R(t,\xi)-S_i(t,\xi)} d\xi + C_i(\bar{t}^e) \tag{5.15}$$

for $(i, j) \in \Upsilon$ where $C_{ij}(\bar{t}^e) = 0$ and $\bar{t}^e = \infty$ if $T^e = \emptyset$.

Proof. Equations (5.10) with $\mu_0 = 0$ yields equation (5.15) with C_i undetermined. □

5.3. **Costate dynamics on the demand constraint.** Costate dynamics on the effective demand constraint can be solved by focusing on "pressure to change employment structure" k^i_{ab}. As mentioned above, this fact is no more than the other side of the coin (literally!) that the model has a demand constraint in the configuration space. Please observe the symmetricity between canonically conjugate coordinates. The key is that the following transformation $\Phi : \Omega \to \omega$ should be applied to the model where $\boldsymbol{\Omega} = (\mathbf{L}_1, \ldots, \mathbf{L}_L, \boldsymbol{\Lambda}_1, \ldots, \boldsymbol{\Lambda}_L)$, $\omega = (\mathbf{l}_1, \ldots, \mathbf{l}_L, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L)$, $\mathbf{L}_i = (L_{i1}, \ldots, L_{iM_i})$, $\boldsymbol{\Lambda}_i = (\Lambda_{i1}, \ldots, \Lambda_{iM_i})$, $\mathbf{l}_i = (l_{i1}, \ldots, l_{iM_i})$ and $\boldsymbol{\lambda}_i = (\lambda_{i1}, \ldots, \lambda_{iM_i})$. Note that one can always choose $j^*(i)$ such that $\sum_a \phi_{ia} f_{ia} k^i_{aj^*(i)} \geq 0$ for all i by taking $j^*(i) = \arg\min_j \lambda_{ij}/f_{ij}$ for given i:

$$\Phi : \quad \begin{pmatrix} l_{i1} \\ \vdots \\ l_{i,j^*(i)} \\ \vdots \\ l_{iM_i} \end{pmatrix} = \begin{pmatrix} L_{i1} \\ \vdots \\ g(t, \mathbf{L}) \\ \vdots \\ L_{iM_i} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda_{i1} \\ \vdots \\ \lambda_{i2} \\ \vdots \\ \lambda_{iM_i} \end{pmatrix} = \begin{pmatrix} \Lambda_{i,j^*(i)} + f_{i,j^*(i)}\Lambda_{i1} \\ \vdots \\ f_{i,j^*(i)}\Lambda_{i,j^*(i)} \\ \vdots \\ \Lambda_{i,j^*(i)} + f_{i,j^*(i)}\Lambda_{iM_i} \end{pmatrix} \quad \text{for all } i.$$

where $g : \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function that satisfies $y(t) - f(\ldots, L_{i,j^*(i)-1}, g(t, \mathbf{L}), L_{i,j^*(i)-1}, \ldots) = L_{i,j^*(i)}$. This is a point transformation and therefore is a special case of canonical transformation. A *canonical transformation* is defined to be a transformation the pull-back of which maps a second order differential form to itself and it is known to be preserving the Hamiltonian function.[10] A *point transformation* is a type of canonical transformation in which the transformation of the configuration subspace is limited to itself.

[10]See Arnol'd (1989) and Ito (1998) for analytical mechanics.

The construction of Φ can be easily observed by the following arguments. Suppose $\lambda_{i1}/f_{i1} = \min_a \lambda_{ia}/f_{ia}$ for given i without loss of generality. We construct a point transformation $\Phi : (\mathbf{L};\boldsymbol{\Lambda}) \to (\mathbf{l};\boldsymbol{\lambda}) = (\varphi(\mathbf{L});\boldsymbol{\lambda})$ such that

$$\varphi^{-1} : \begin{pmatrix} L_{i1} \\ L_{i2} \\ \vdots \\ L_{iM_i} \end{pmatrix} = \begin{pmatrix} y(t) - f(\mathbf{l}) \\ l_{i2} \\ \vdots \\ l_{iM_i} \end{pmatrix} \qquad \text{or equivalently} \qquad \varphi : \begin{pmatrix} l_{i1} \\ l_{i2} \\ \vdots \\ l_{iM_i} \end{pmatrix} = \begin{pmatrix} g(t,\mathbf{L}) \\ L_{i2} \\ \vdots \\ L_{iM_i} \end{pmatrix}.$$

For it to be a point transformation, Hamiltonian must be invariant with this except for the "time-variant" term. Therefore,

$$\begin{pmatrix} \Lambda_{i1} \\ \Lambda_{i2} \\ \vdots \\ \Lambda_{iM_i} \end{pmatrix} = {}^t\varphi_L \begin{pmatrix} \lambda_{i1} \\ \lambda_{i2} \\ \vdots \\ \lambda_{iM_i} \end{pmatrix} = \left(\begin{array}{c|ccc} -\frac{1}{f_1} & 0 & \cdots & 0 \\ \hline -\frac{f_2}{f_1} & 1 & & O \\ \vdots & & \ddots & \\ -\frac{f_{M_i}}{f_1} & O & & 1 \end{array}\right) \begin{pmatrix} \lambda_{i1} \\ \lambda_{i2} \\ \vdots \\ \lambda_{iM_i} \end{pmatrix} = \begin{pmatrix} -\frac{\lambda_1}{f_1} \\ \lambda_2 - \frac{f_2}{f_1}\lambda_1 \\ \vdots \\ \lambda_{M_i} - \frac{f_{M_i}}{f_1}\lambda_1 \end{pmatrix}$$

must hold, which implies

$$\begin{pmatrix} \lambda_{i1} \\ \lambda_{i2} \\ \vdots \\ \lambda_{iM_i} \end{pmatrix} = \begin{pmatrix} -f_1\Lambda_1 \\ \Lambda_2 + f_2\Lambda_1 \\ \vdots \\ \Lambda_{M_i} + f_{M_i}\Lambda_1 \end{pmatrix}.$$

Note that, with this choice of $j^*(i)$, it becomes possible to make $\sum_a \phi_{ia}\Lambda_{ia} \geq 0$ so that the transformation does not conflict with the limitation that $\kappa'^{-1}(\cdot)$ is defined only on domain $\mathbb{R}_+$.

The Hamiltonian K on the new coordinates is given by

$$K(t,L,\Lambda) = H(t,\Phi(t,L,\Lambda)) - \left\langle \varphi_t, \left({}^t\varphi_L\right)^{-1}\Lambda \right\rangle$$

which simplifies to

$$\begin{aligned} K = {} & f\Big(\ldots, L_{i,j^*-1}, g(t,L), L_{i,j^*+1}, \ldots\Big) - c\Big(\ldots, L_{i,j^*-1}, g(t,L), L_{i,j^*+1}, \ldots\Big) \\ & - \sum_i \kappa_i\left(m_i\left(\sum_{a\neq j^*}\phi_{ia}\Lambda_{ia}\right)\right) + \sum_i\sum_{a\neq j^*}\Lambda_{ia}\left(\phi_{ia} m\left(\sum_{b\neq j^*}\phi_{ib}\Lambda_{ib}\right) - \sigma_{ia}L_{ia}\right) \end{aligned} \tag{5.16}$$

where $m(\cdot) := \kappa'^{-1}(\cdot)$. The equation does not contain Λ_{j^*}, showing that L_{j^*} is an cyclic coordinate. This is an *energy surface* on which a path is restricted. When $L = 1$ and $M_1 = 2$, it implicitly but fully characterizes the solution. Note that λ_{j^*} is indeterminant on the demand surface.

5.4. **Costate discontinuity on junction points.**

Entering condition to the demand constraint. Let $C \subset \mathbb{R}^M$ be a configuration space. Define an *entering time* $t^e \in \mathbb{R}$ to a state constraint surface $\mathcal{B} \subset C$ such that the costate variable μ_0 adjoint to the state constraint $\mathcal{B}$ yields $\mu_0(t^e) = \mu_0(t^e - \varepsilon) = 0$ and $\mu_0(t^e + \varepsilon) > 0$ for any arbitrarily small $\varepsilon > 0$. Let $z(t) : \mathbb{R} \to C$ be a path, i.e. a trajectory projected onto the configuration space. Then, $z(t^e)$ is called an *entering point*. Similarly, *leaving time* $t^l \in \mathbb{R}$ from a state constraint $\mathcal{B}$ is defined to be $\mu_0(t^l) = \mu_0(t^l + \varepsilon) = 0$ and $\mu_0(t^l - \varepsilon) > 0$ for any arbitrarily small $\varepsilon > 0$. $z(t^l)$ is called a *leaving point*. Denote a set of all entering times by T^e and and a set of all leaving time by T^l. We also call $t^j \in T^e \cap T^l$ a junction time and $z(t^j)$ a junction point. In general, costate variables are time-discontinuous either at entering points to or at leaving points from the state constraint. And generally, it is unknown whether the discontinuity occurs at entering points or at leaving points (Bryson et al. (1963)). This is due to the fact that a state constraint imposes restriction on controls in its derivative form. Its initial value can be be specified either by a initial point or a terminal point. When costate variables are continuous at entering time and discontinuous at leaving time, costate variables are interpreted as normals to neighborhood of the optimal trajectory on the limiting surface when dynamics is controlled by optimal control after the entering time. However, in our problem, we will find out that discontinuity should occur only at *entering points*. In other words, costate variables in this problem are interpreted as normals to neighborhood of the optimal trajectory on the limiting surface when the neighborhood is a transformation of the one at the *leaving time* and dynamics is controlled backward by optimal control tracing back from the leaving time. This is due to the forward-looking property of economic dynamics.

At a junction point $t \in T^e \cap T^l$,

$$\lambda_{ia}^- = \lambda_{ia}^+ + \rho f_{ia} \tag{5.17}$$

$$H^- = H^+ + \rho \dot{y} \tag{5.18}$$

where ρ is an indeterminate variable adjoint to the state constraint $y - f(\mathbf{l}) = 0$ and, for any variable A, we denote $A^- := \lim_{t \uparrow t^j} A$, $A^+ := \lim_{t \downarrow t^j} A$. From (5.17),

$$\rho = \frac{\lambda_{ia}^- - \lambda_{ia}^+}{f_{ia}}$$

for all i and a, which implies

$$\Delta k_{ab}^i = \frac{\Delta \lambda_{ia}}{f_{ia}} - \frac{\Delta \lambda_{ib}}{f_{ib}} = 0 \tag{5.19}$$

for all i, a and b such that $a \neq b$. Namely, the jump at junction points occurs along contour lines in Figure 5.1 so that k_{ab}^i does not change.

Noting that $\pi^+ = \pi^-$, we obtain, from (5.18),

$$\sum_{i,j} \lambda_{ij}^- \left(\phi_{ij} m_i^- - \sigma_{ij} l_{ij} - x_{ij}^-\right) - \sum_i \kappa_i \left(m_i^-\right) = \sum_{i,j} \lambda_{ij}^+ \left(\phi_{ij} m_i^+ - \sigma_{ij} l_{ij} - x_{ij}^+\right) - \sum_i \kappa_i \left(m_i^+\right) + \rho \dot{y}$$

Using (5.17), it turns out

$$\sum_i \kappa_i \left(m_i^+\right) - \sum_i \kappa_i \left(m_i^-\right) - \sum_{i,j} \left(\phi_{ij} \lambda_{ij}^-\right) \left(m_i^+ - m_i^-\right) + \sum_{i,j} \left(x_{ij}^+ - x_{ij}^-\right) \lambda_{ij}^- = \begin{cases} 0 & \text{if } t \in T^e \\ \rho \left(\dot{y} - f_1 \dot{l}_1^+ - f_2 \dot{l}_2^+\right) & \text{if } t \in T^l \end{cases} \tag{5.20}$$

Or, the same relation can be expressed as

$$\text{(5.20')}\qquad \sum_i \kappa\left(m^-\right) - \sum_i \kappa\left(m^+\right) - \sum_{i,j}\left(\phi_{ij}\lambda_{ij}^-\right)\left(m^- - m^+\right) + \sum_{i,j}\left(x_{ij}^- - x_{ij}^+\right)\lambda_{ij}^- = \begin{cases}\rho\left(\dot{y} - f_1\dot{l}_1^- - f_2\dot{l}_2^-\right) & \text{if } t \in T^e\\ 0 & \text{if } t \in T^l\end{cases}$$

Proposition 5.5. *At both entering and leaving points, m_i is continuous at $m_i = \bar{m}_i$. x_{ij} is continuous at $x_{ij} = 0$ at leaving points and also at entering points when $\dot{y} \geq -\sum_{i,j}\sigma_{ij}l_{ij}$. If $\dot{y} < -\sum_{i,j}\sigma_{ij}l_{ij}$, $x_{ij}^+ > 0$ for some (i, j) possibly showing discontinuity.*

Proof. When $t \in T^e$, $\lambda_{ij}^- \geq 0$ for $\forall i, j$. If $\lambda_{ij}^- < 0$ for some i and j, then $x_{ij}^- = X$, which implies $\sum_{i,j} f_{ij}\dot{l}_{ij}^- \ll \dot{y}$, violating the entering condition $\sum_{i,j} f_{ij}\dot{l}_{ij}^- > \dot{y}$. First, suppose $\lambda_{ij}^- > 0$ for some i and j. Then, (5.20) becomes

$$\sum_i \kappa_i\left(m_i^+\right) = \sum_i \kappa_i\left(m_i^-\right) + \sum_i \kappa_i'\left(m^-\right)\left(m_i^+ - m_i^-\right) - \sum_{i,j} x_{ij}^+\lambda_{ij}^-$$

when $t \in T^e$. However, since κ_i is a convex function and $x \geq 0$, the above relation is only possible when $m_i^+ = m_i^-$ and $x_{ij}^+ = 0$ for all i and j. On the other hand, if $\lambda_{ij}^- = 0$ for all i and j, (5.20) yields $\sum_i \kappa_i(m_i^+) = \sum_i \kappa_i(m_i^-) = 0$ and again m_i is continuous at zero for all i. In this case, $k_{ab}^i(t^e) = \lambda_{ia}^+/f_{ia} - \lambda_{ib}^+/f_{ib} = \lambda_{ia}^-/f_{ia} - \lambda_{ib}^-/f_{ib} = 0$ for all i, a, b which implies $x_{ij} = 0$ for all i, j as far as $\dot{y} \geq -\sum_{i,j}\sigma_{ij}l_{ij}$ by Proposition 5.3. If $\dot{y} < -\sum_{i,j}\sigma_{ij}l_{ij}$, some of x_{ij} are strictly positive according to (2) of Proposition 5.3.

At leaving points, (5.20') becomes

$$\sum_i \kappa_i\left(m_i^-\right) = \sum_i \kappa_i\left(m_i^+\right) + \sum_i \kappa_i'\left(m_i^+\right)\left(m_i^- - m_i^+\right) - \sum_{i,j} x_{ij}^-\lambda_{ij}^+$$

when $t \in T^l$. However, since κ is a convex function and $x \geq 0$, the above relation is only possible when $m_i^+ = m_i^-$ and $x_{ij}^- = 0$ for all i and j.

Since $x_{ij} = 0$ for all i, j, $m_i = \bar{m}_i$ for all i, for both entering and leaving points from Proposition 5.3, □

Corollary 5.6. *An entering point locates on a closed interval $\sum_j\left(\phi_{ij}\lambda_{ij}^-\right) = \kappa'(\bar{m}_i)$ where $\lambda_{ij}^- \geq 0$ Moreover, if $\dot{y} \geq 0$, $\lambda_{ij^*} = 0$ for $j^* = \arg\min_j \lambda_j$.*

Proof. Since $x_{ij}^+ = 0$ from Proposition 5.5, k_{ab}^i must be such that $\sum_a \phi_{ia}f_{ia}k_{aj}^i \leq \kappa_i'(\bar{m}_i)$ at the entering points (Proposition 5.3 (1)). Since m_i is continuous at $m_i = \bar{m}_i$ from Proposition 5.5, the entering point must be located on a closed segment $\sum_j \phi_{ij}\lambda_{ij}^- = \kappa_i'(\bar{m}_i)$ in the conjugate space with the condition $\sum_a \phi_{ia}f_{ia}k_{aj}^i \leq \kappa_i'(\bar{m}_i)$ for all j.

Suppose that the entering point was an inner point of $\{\boldsymbol{\lambda} : \sum_a \phi_{ia}f_{ia}k_{aj}^i \leq \kappa_i'(\bar{m}_i)\}$. Then, by the continuity of $\boldsymbol{\lambda}$, optimal control at $t = t^e + \varepsilon$ is unchanged: $m_i(t^e + \varepsilon) = \bar{m}_i(t^e + \varepsilon)$ and $x_{ij} = 0$. Since $\dot{y} \geq 0$, it implies that $m_i(t^e + \varepsilon) > \bar{m}_i(t^e)$. If we hypothetically assume that the path is unbounded after the entering time, $\kappa_i'(m_i(t^e + \varepsilon)) = \sum_j \phi_{ij}\lambda_{ij}(t^e + \varepsilon) < \sum_j \phi_{ij}\lambda_{ij}(t^e) = \kappa_i'(m_i(t^e))$, since $\dot{\lambda}_{ij} < 0$. It implies that $m_i(t^e + \varepsilon) < \bar{m}_i(t^e)$. It implies that the constraint is actually unbinding immediately after entering time. Therefore, the entering point must be located on the border of $\{\boldsymbol{\lambda} : \sum_a \phi_{ia}f_{ia}k_{aj}^i \leq \kappa_i'(\bar{m}_i)\}$. It implies that the equality must be set for $j^* = \arg\min_j \lambda_j$. □

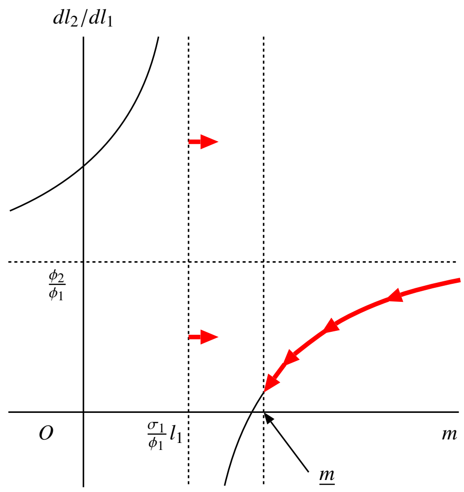


FIGURE 5.3. Change of the slope of displacement vector when $m \downarrow \underline{m}$

Note that, since $\rho = -\lambda^+_{ij}/f_{ij}$, the costate variables are generally discontinuous at the entering points. From Proposition 5.5, entering points locate in domain $\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa'_i(\bar{m}_i)$. It implies that the entering to the demand constraint must be "smooth" in the configuration space. Namely, growth of labor must slow down as employment approaches to the demand constraint. The next proposition shows that the slowdown accompanies revolution in the configuration subspace between different *undeclarable* types.

Proposition 5.7. *If an entering point is located in domain* $\{(l_{ia}, l_{ib}) : (\sigma_{ia}/\phi_{ia})l_{ia} < (\sigma_{ib}/\phi_{ib})l_{ib}\}$, *the path prior to entering shows clockwise revolution. If entering occurs in domain* $\{(l_{ia}, l_{ib}) : (\sigma_{ia}/\phi_{ia})l_{ia} > (\sigma_{ib}/\phi_{ib})l_{ib}\}$, *the path prior to entering shows counterclockwise revolution.*

Proof. The displacement vector when the state constraint is unbinding is given by

$$(dl_{ia}, dl_{ib}) = dl_{ia}\left(1, \frac{\dot{l}^-_{ib}}{\dot{l}^-_{ia}}\right) = dl_{ia}\left(1, \frac{\phi_{ib}}{\phi_{ia}}\left(1 + \frac{\frac{\sigma_{ia}}{\phi_{ia}}l_{ia} - \frac{\sigma_{ib}}{\phi_{ib}}l_{ib}}{m - \frac{\sigma_{ia}}{\phi_{ia}}l_{ia}}\right)\right)$$

Since $\dot{l}_{ia} > 0$ before entering, if $(\sigma_{ia}/\phi_{ia})l_{ia} < (\sigma_{ib}/\phi_{ib})l_{ib}$, then $d^2 l_{ib}/dl_{ia}dt < 0$. This can be observed by Figure 5.3. The case for $(\sigma_{ia}/\phi_{ia})l_{ia} > (\sigma_{ib}/\phi_{ib})l_{ib}$ is obtained by symmetricity and $d^2 l_{ib}/dl_{ia}dt > 0$. □

The behavior between different *declarable* labor types prior to entering is more complex. In such a case, the displacement vector is characterized by $dl^-_{ia}/dl^-_{jb} = (\phi_{ia}m_i - \sigma_{ia}l_{ia})/(\phi_{jb}m_j - \sigma_{jb}l_{jb})$ the movement of which is also affected by the relative convergence speed of m_i and m_j, not only relative size of labor.

Leaving condition from the demand constraint. Leaving points exist in the *interior* of $\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa'_i(\bar{m}_i)$. Note that leaving from the demand constraint never occurs so far as $\dot{y} \leq 0$. Leaving occurs when catchup to the growth of demand becomes too costly in terms of accompanying vacancy cost.

As $\dot{y}$ becomes too large, it becomes suboptimal to stick to the surface of the demand constraint. What happens is that as $\dot{y}$ grows, the band $B \geq k \geq C$ ($\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa'_i(\bar{m}_i)$) in Figure 5.1 widens while the width of other bands are kept constant. It implies that for given k, it becomes more likely to fall in the domain $B \geq k \geq C$. Unless the value of $\dot{y}$ is such that corresponding optimal control keeps the state variables *exactly* on the surface of the demand constraint, as soon as k falls in the domain $B \geq k \geq C$, the state variables leaves the demand constraint. The leaving is more likely to happen if $|k|$ is small.

Proposition 5.8. *Leaving points locate in the interior domain* $\sum_a \phi_{ia} f_{ia} k^i_{aj} < \kappa'_i(\bar{m}_i)$ *and* λ*'s are continuous on those points.*

Proof. Similar to Proposition 5.6, $\sum_a \phi_{ia} f_{ia} k^i_{aj} \leq \kappa'_i(\bar{m}_i)$ must hold. However, if $\sum_a \phi_{ia} f_{ia} k^i_{aj} = \kappa'_i(\bar{m}_i)$, $\lambda^+_i = 0$ for some i. This is impossible. Therefore, leaving points locate in interior domain $\sum_a \phi_{ia} f_{ia} k^i_{aj} < \kappa'_i(\bar{m}_i)$.

Next, suppose that λ's are discontinuous at leaving points, i.e. $\lambda^+_{ij} \neq \lambda^-_{ij}$ for all i, j. Since $\mu_0 = 0$ at leaving points, $\kappa'_i(m^+_i) = \sum_j \phi_{ij} \lambda^+_{ij}$ and $\kappa'_i(m^-_i) = \sum_j \phi_{ij} \lambda^-_{ij}$, which implies that $m^+_i < m^-_i$ (opposite inequality is impossible because the path is leaving below the constraint). Now, since λ's are continuous except junction points, we can take sufficiently small $\varepsilon > 0$ such that states at time $t^l - \varepsilon$ remain interior of domain III and

$$\kappa'_i(m_i(t^l - \varepsilon)) = \sum_j \phi_{ij} \lambda_{ij}(t^l - \varepsilon) - \mu_0(t^l - \varepsilon) \sum_j \phi_{ij} f_{ij} > \kappa'_i(m^+_i(t^l))$$

where $\mu_0(t^l - \varepsilon) > 0$ and converges to zero as $\varepsilon \to 0$. Therefore, we have

$$\begin{aligned} m_i(t^l - \varepsilon) &> m^+_i(t^l) \\ x_{ij}(t^l - \varepsilon) &= x_{ij}(t^l) = 0 \qquad \forall i, j. \end{aligned}$$

Then,

$$\begin{aligned} \dot{y}(t^l - \varepsilon) &= \sum_i \left[\left(\sum_j \phi_{ij} f_{ij} \right) m_i(t^l - \varepsilon) - \sum_j \sigma_{ij} f_{ij} l_{ij}(t^l - \varepsilon) \right] \\ &> \sum_i \left[\left(\sum_j \phi_{ij} f_{ij} \right) m^+_i(t^l) - \sum_j \sigma_{ij} f_{ij} l_{ij}(t^l - \varepsilon) \right] \\ &\to \sum_i \left[\left(\sum_j \phi_{ij} f_{ij} \right) m^+_i(t^l) - \sum_j \sigma_{ij} f_{ij} l_{ij}(t^l) \right] = \dot{y}(t^l) \end{aligned}$$

as $\varepsilon \downarrow 0$. Namely, $\dot{y}^- > \dot{y}^+$ at leaving points. This contradicts the assumption that $\dot{y}$ is continuous. Therefore, λ's must be continuous at leaving points. □

6. Rationale of Wage Function

λ is an influence function of l on the value of the optimand *in the maximization problem for a firm.* To justify a wage function derived in Section 3 and 4, however, we need to know the marginal impact of change in l on

$$F(l) = \int_t^\infty \left(f - \sum_i \kappa_i \right) e^{-R} d\xi$$

instead of J, *when l follows the optimal employment path for a firm.* Actually, it can be shown that a new "influence" function of l on F can be constructed based on the derivation of λ. Let the new influence function denoted by λ^*. The above equation can be rewritten

$$F(l) = \int_t^\infty \left(H + c + \dot{\lambda}^* \cdot l\right) e^{-R} d\xi - \lambda^*(\infty) \cdot l(\infty) + \lambda^*(t) \cdot l(t) \tag{6.1}$$

using the new costate variable λ^*. Taking total derivative[11]

$$\delta F = \int_t^\infty \left[\left(\frac{\partial H}{\partial l} + \frac{\partial c}{\partial l} + \dot{\lambda}^*\right)\delta l(\xi) + \frac{\partial H}{\partial u}\delta u(\xi)\right] dt - \lambda^*(\infty) \cdot \delta l(\infty) + \lambda^*(t) \cdot \delta l(t)$$

where $u = (m, x_1, x_2)$. We want to set λ so that we can neglect the effect of $\delta l(\xi)$ $(t < \xi < \infty)$ on δJ. Then, dynamics of λ^* should be given by

$$\dot{\lambda}^* = -\frac{\partial H}{\partial l} - \frac{\partial c}{\partial l} = \dot{\lambda} - \frac{\partial c}{\partial l}$$

and the new $\lambda^*(t)$ which follows the above dynamics is the "influence" of l upon the payoff of the total coalition. It implies

$$\lambda^*_{ij} = \int_t^{t^e} f_{ij}\, e^{-\int (r+\sigma_{ij})} ds + C \tag{6.2}$$

from $\mu_0 = 0$ in equation (5.10). When distribution between workers and a firm is bargained, marginal impact of *decrease* in coalition matters. When an agent leaves the coalition, bounded demand becomes unbinding. Therefore, regardless whether demand constraint is binding, (6.2) is pertinent to the problem.

Theorem 6.1. *F has the property that* $\partial F/\partial l_{ij}- > 0$ *and* $\partial^2 F/\partial l_{ij}^2- < 0$ *for all* (i, j)*.*

Proof. From equation (6.2),

$$\frac{\partial \lambda^*_{ij}}{\partial l_{ij}(t)} = \int_t^{t^e} \sum_h \frac{\partial^2 f(s)}{\partial l_{ih}(s)^2} \frac{\partial l_{ih}(s)}{\partial l_{ij}(t)} e^{-\int (r+\sigma)} ds + \frac{\partial t^e}{\partial l_{ij}(t)} \frac{\partial f(t)}{\partial l_{ij}(t)} < 0$$

for all $(i, j) \in \Upsilon$ in either case that the demand constraint is binding or not binding. Note that after the change of $l_{ij}(t)$, for all $s > t$, $\partial l_{ij}(s)/\partial l_{ij}(t)- > 0$ and $dl_{ih}(s)/\partial l_{ij}(t)- < 0$ where $j \neq h$ since $x = 0$ is optimal control after the shock. Also, $\partial t^e/\partial l(t) < 0$. Since $\partial^2 F/\partial l_{ij}(t)^2 = \partial \lambda^*_{ij}/\partial l_{ij}(t)$, it implies that function F is $\partial F/\partial l_{ij} > 0$ and $\partial^2 F/\partial l_{ij}^2 < 0$. □

Theorem 6.1 shows that the present value of the coalition F satisfies the condition of Definition 3.1. Namely, we proved that the fundamental game between workers and a firm presented in Section 3 is consistent with the model. The next theorem completes the argument that there will be excess demand for labor if the demand constraint is unbinding which is the source of the (modified) principle of effective demand. Since increase of labor always amplifies profit of firm, it is always willing to accommodate additional potential demand as far as it is smaller than the unbounded steady state level.

Theorem 6.2. *If the demand constraint is unbinding and* $\mathbf{l}$ *is smaller than the unbounded steady state,* $dJ/d\mathbf{l} > 0$*. Moreover, if* $\partial F/\partial t \partial l_{ij} - \dot{U}_i \leq 0$*, then* $w_{ij}(t) < \partial f(t)/\partial l_{ij}$ *for all* $(i, j) \in \Upsilon$ *and* t*.*

[11]The differential operator is denoted by δ to distinguish it from the infinitesimal increment in integration.

Proof. The first statement is obvious from the fact that $\lambda_{ij} > 0$ for all i, j when the demand constraint is unbinding and $\mathbf{l}$ is smaller than the unbounded steady state. The second statement comes from the following. From $\partial F/\partial l_i = \lambda_i^*$, $\dot{\lambda}_{ij}^* = -f_{ij} + (r + \sigma_{ij})\lambda_{ij}^*$. Equation (4.5) yields

$$\begin{aligned} f_{ij} - w_{ij} &= \left(w_{ij} - b_i\right) + \sigma_i \lambda_{ij}^* + \mu_i \int_t^\infty \mathbf{E}_h \left[w_{ih} - b_i\right] e^{-\int \alpha} ds - 2\sigma_i (E_{ij} - U_i) \\ &= \left(w_{ij} - b_i\right) + \mu_i \int_t^\infty \mathbf{E}_h \left[w_{ih} - b_i\right] e^{-\int \alpha_i} ds + \sigma_i U_i \\ &> 0 \end{aligned}$$

where the second line is obtained using $E_{ij} - U_i = (\lambda_{ij}^* - U_i)/2$ and the last inequality comes from Proposition 4.2 and Proposition 4.3. □

7. Steady State on the Demand Surface

The model allows for an analysis of a perpetually moving economy. However, to settle down the endpoint of costate variables, it is convenient to analyze the steady state. The previous analyses showed that unless there is coordinated expectation among economic agents which persists for infinite length of time, the economy will not reach to an unbounded steady state. However, an economy can find out a steady state on a binding demand constraint with reasonable size of demand. There is at least one candidate for it —— if we can think of an economy which has stabilized effective demand, it will find a steady state with the current level of output. Suppose $\dot{y} = 0$. Then, it is found out that strictly positive amount of rejection of job application will occur at the steady state unless parameters satisfy a relation which has a zero measure in the parameter space.

It will be shown that a bounded steady state maximizes the profit when initial state of labor can be chosen directly and when labor —which is an asset— is discounted by interest rate. Consider the following problem.

$$\text{(P')} \qquad \max_{l,m,x} \left\{ f(\mathbf{l}) - w(\mathbf{l}) \cdot \mathbf{l} - \sum_{i=1}^{L} \kappa_i(m_i) \right\}$$

subject to

$$\text{(5.1')} \qquad \phi_{ij} m_i = (r + \sigma_{ij}) l_{ij} + x_{ij}, \ \forall i, j$$

$$\text{(5.4')} \qquad y = f(\mathbf{l})$$

Theorem 7.1. *Steady-state solution of problem (P) is equivalent to the solution of (P').*

Proof. The optimality condition of the problem (P') is given by

$$\text{(5.7')} \qquad \kappa_i'(m_i) = \sum_j \phi_{ij} \hat{\lambda}_{ij}$$

$$\text{(5.10')} \qquad \hat{\lambda}_{ij} = \frac{f_{ij} - c_{ij}}{r + \sigma_{ij}} - \hat{\mu}_0 \frac{f_{ij}}{r + \sigma_{ij}}$$

(5.11')
$$x_{ij} = \begin{cases} 0 & \text{if } \hat{\lambda}_{ij} > 0 \\ X & \text{if } \hat{\lambda}_{ij} < 0 \end{cases}$$

and the constraints where $\hat{\lambda}_{ij}$ and $\hat{\mu}_0$ are costate variables adjoint to equations (5.1') and (5.4'), respectively. From (5.7') and (5.10'),

(7.1)
$$\hat{\mu}_0 = \frac{\sum_j \frac{\phi_{ij}}{r+\sigma_{ij}}(f_{ij} - c_{ij}) - \kappa_i'(m_i)}{\sum_j \frac{\phi_{ij}}{r+\sigma_{ij}} f_{ij}}$$

Since X is arbitrarily large and therefore the steady state condition for l_{ij} does not hold when $x_{ij} = X$, $\hat{\lambda}_{ij} < 0$ is impossible for all i. Thus, $\hat{\lambda}_{ij} > 0$ or $\hat{\lambda}_{ij} = 0$. If there exist (i, j) such that $\hat{\lambda}_{ij} = 0$, then for such (i, j)'s

(7.2)
$$\hat{\mu}_0 = \frac{f_{ij} - c_{ij}}{f_{ij}} \qquad \forall (i, j),\ \hat{\lambda}_{ij} = 0$$

and for other (i, j)'s such that $\hat{\lambda}_{ij} > 0$,

(7.3)
$$x_{ij} = 0 \qquad \forall (i, j),\ \hat{\lambda}_{ij} > 0$$

holds. Then, the solution is completely characterized by (5.1'), (5.4'), (7.1), (7.2) and (7.3).

On the other hand, the bounded steady state solution to the original problem (P) is given by imposing steady state condition $\dot{l} = \dot{\lambda} = \dot{f}_{ij} = 0$ to each optimal condition. Imposing it on (5.1) and (5.4) obtains the same condition as (5.1') and (5.4'). From (5.10) and the steady state conditions,

(5.10")
$$\lambda_{ij} = \frac{f_{ij} - c_{ij}}{r + \sigma_{ij}} + \mu_0 \frac{\sigma_{ij} f_{ij}}{r + \sigma_{ij}}.$$

Substituting this to (5.7) derives

(7.1')
$$\mu_0 = \frac{\sum_j \frac{\phi_{ij}}{r+\sigma_{ij}}(f_{ij} - c_{ij}) - \kappa_i'(m_i)}{r \sum_j \frac{\phi_{ij}}{r+\sigma_{ij}} f_{ij}},$$

which is equivalent to (7.1) if we define $\hat{\mu}_0 = r\mu_0$. From (5.10') and (5.10"), $\lambda_{ij} = \hat{\lambda}_{ij} + \mu_0 f_{ij}$, which results in equivalence relation between

$$x_{ij} = \begin{cases} 0 & \text{if } A_{ij} > 0 \\ [0, X] & \text{if } A_{ij} = 0 \end{cases} \text{ in problem (P)} \iff x_{ij} = \begin{cases} 0 & \text{if } \hat{\lambda}_{ij} > 0 \\ [0, X] & \text{if } \hat{\lambda}_{ij} = 0 \end{cases} \text{ in problem (P').}$$

All of the above equivalences show that problem (P') is equivalent to problem (P). □

The next theorem shows that, in general, the point in which long-run profit is maximized does not coincide with the point in which a bounded steady state is achieved with no-firing. It means that either dismissal or rejection of job application will occur at a bounded steady state.

Theorem 7.2. *If $\sum_{i=1}^{L} M_i \geq 2$, the set of parameters (ϕ, σ) that brings $x_{ij} = 0$ for all (i, j) at steady state has zero measure in the parameter space for given f and κ.*

Proof. From Theorem 7.1, the proposition can be proved via problem (P'). x which appears in (P') can be viewed as a slack variable substituting equality of equation (5.1') with inequality. Namely, it is equivalent to the following problem:

$$\text{(P'')} \qquad \min_{l,m} \left\{ w(\mathbf{l}) \cdot \mathbf{l} + \sum_{i=1}^{L} \kappa_i(m_i) \right\}$$

subject to

$$\text{(5.1'')} \qquad \phi_{ij} m_i \geq \left(r + \sigma_{ij}\right) l_{ij}, \quad \forall i, j$$

$$\text{(5.4')} \qquad y = f(\mathbf{l})$$

Obviously, m_i maximizes the maximand when it is set to $m_i = \min_j\{(r + \sigma_{ij}) l_{ij} / \phi_{ij}\}$ in equation (5.1''). Maximization on $\mathbf{l}$ with this condition completely determines solution for $\mathbf{l}$. However, in general,

$$\frac{r + \sigma_{ij}}{\phi_{ij}} l_{ij} \neq \frac{r + \sigma_{ij'}}{\phi_{ij'}} l_{ij'}$$

for any $j' \neq j$, making $x_{ij'} > 0$ for any j' such that $j' \neq \arg\min_j\{(r + \sigma_{ij}) l_{ij} / \phi_{ij}\}$. Even when the condition

$$\text{(7.4)} \qquad \frac{r + \sigma_{ij}}{\phi_{ij}} l_{ij} = \frac{r + \sigma_{ij'}}{\phi_{ij'}} l_{ij'}$$

for all j, j', i holds, it fails to hold once any small perturbation is added on one of r,σ or ϕ keeping other parameters. Namely, a set of parameters which satisfies (7.4) does not contain inner points, which implies that it has zero measure in the parameter space when $\sum_{i=1}^{L} M_i \geq 2$. □

The above theorem shows that dismissal or rejection of application *generically* occurs at least in one of the labor types not only in transition on the demand constraint surface but also at steady state, when there exist more than two labor types in the economy.

Figure 7.1 shows typical dynamics toward steady state when $L = 1$ and $M_1 = 2$. Paths starting from initial points A_1, A_2 and A_3 converges to a steady state C when steady state demand level is y_1. The paths starting from A_1 and A_2 enter the demand surface with clockwise motion whereas the one starting from A_3 shows counterclockwise movement. If demand unexpectedly shifts up to $y = y_2$ when the state is in neighborhood of C, the path starts to move toward the new demand surface and after counterclockwise entering, it continues with zero dismissal until it crosses a line which passes through the origin. After crossing over the line, it starts to dismiss type 1 workers and converges to a new steady state D.

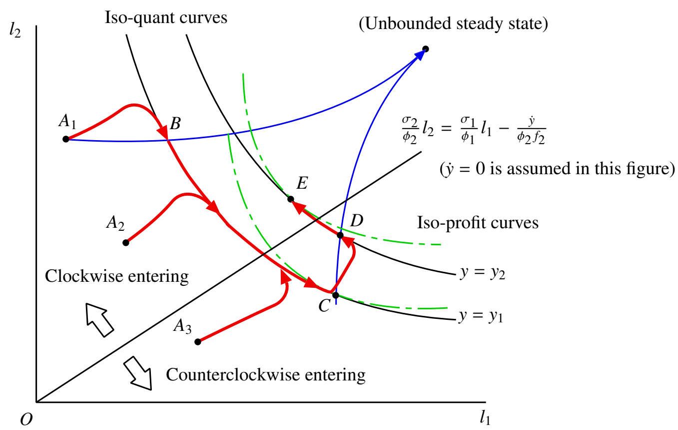


FIGURE 7.1. Unexpected shift of the demand constraint

8. Concluding Remarks

This paper showed that if there is search friction representable by a convex vacancy cost function ——however small cost for a given amount of hiring——, the economy obeys the effective demand principle. Wage rate is always smaller than marginal productivity, and a direct attempt to lower wage rate will not remove unemployment, as the old Keynesian arguments suggest. It should be noted that any kinds of sticky price is not assumed in this model. The existence of convex vacancy cost prohibits convergence to an unbounded steady state, or an *equilibrium in the long run*, without persistent coordination of expectation. Wage rate is flexible reflecting redundant resources in the labor market.

In search models, profit of a firm is strictly positive even when the commodity market is competitive. The fact that an entrepreneur earns non-zero profit and that he has massive power in bargaining as suggested in this paper raises a fundamental question that who really is the "entrepreneur". The question cannot be neglected when one undertakes to specify the demand side explicitly because it affects distribution of income. There can be two most straightforward but extreme ways of extension: one is to assume that income level has no impact on consumption behavior. The other is to assume that there are two classes, workers and entrepreneurs in a Kaldorian way. The latter literally assumes that the entrepreneur (and his successor) embodies all the knowledge needed to manage firm and it will never be transferred to workers. It enables the analysis of distribution impact in a simple but extreme way. In the model presented in this paper, the equilibrium condition $V = 0$ does not have an explicit role. A similar condition is still effective, although now V is a function of potential demand as well as other related variables. Therefore, it is not the ultimate condition which determines the number of firms. Rather, the size of potential demand limits it. The decision whether a potential entrepreneur should start a business depends on *future expectation* of the marketability of his own technology and business. It should be immediately added that the model naturally allows for product differentiation in a competitive environment, since allocation of effective demand is inevitable.

The model only determines relative wages at each moment as the wage bargaining is based on rational expectation on both sides. With a help of a consumer theory which contains intertemporal decisions, it determines relative prices between output goods at different moments. Note that there is no built-in mechanism which brings the economy back to a *natural level of output* nor *natural rate of unemployment.* In such an economy, inflation can be a non-monetary phenomenon. Also, the widely observed long-run stability of unemployment rate might be attributed to more long-run factors instead of natural rate, such as industrial structure. It should be noted that autarky, or agriculture in a broader sense, always possesses a special role as a fall-back option for workers, especially in a traditional society.

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E-mail address: yokota@res.otaru-uc.ac.jp