Repeated Moral Hazard with Worker Mobility via Directed On-the-Job Search

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Abstract

This paper studies a repeated moral hazard problem in a general equilibrium framework. I develop a model of dynamic employment contracts by integrating an optimal contracting problem into an equilibrium search framework. The proposed framework enables us to analyze the interaction between the contracting problem and endogenously evolving outside environment via worker mobility, and I characterize the optimal long-term wage contract as well as the optimal incentive compatible effort-tenure profile. The optimal contract exhibits an increasing wage-tenure profile for two reasons: 1) it induces the workers to stay more likely in their current contracts, and 2) it induces the workers to make efforts when the current up-front wages cannot. The optimal incentive-compatible effort has also an increasing profile due to an interaction between 1) the workers' fear of loosing their jobs, and 2) their incentive to obtain better outside offers. I then show the existence of an equilibrium. The equilibrium inherits the "block recursivity" developed by Shi (2008) and Menzio and Shi (2008); that is, individuals' optimal decisions and optimal contracts are independent of the distribution of workers.

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1 Introduction

Structures of long-term contractual relationship in the economy are intrinsically related to the mobility of the contracting party. For example, worker motivation inside a firm changes over time depending on alternative employment opportunities outside the firm. Workers may find outside opportunities more appealing and choose to leave their current employer. This mobility, in turn, affects long-term contractual structure inside the firm. Understanding this interaction between the structure of long-term contracts and mobility of workers within a dynamic environment is the central issue of this paper.

I study a repeated moral hazard problem in a dynamic general equilibrium framework. I develop a model of long-term employment contracting by integrating a moral hazard problem into an equilibrium search framework. The proposed framework enables us to examine properties of the optimal long-term contract, especially a wage-tenure profile, in relation to worker mobility via directed on-the-job search and endogenously evolving outside environment. It is also possible to investigate how workers' efforts evolve over tenure; i.e., are senior workers make more effort or less than junior workers?

In my model, there are risk-neutral ex-ante homogeneous firms and risk-averse ex-ante homogeneous workers in a labor market. Firms offer contracts with different *values* each other to attract workers, and each value forms a submarket. Workers are allowed to search for a new contract both on- and off-the-job. They direct their search for a job, i.e., choose which submarket to enter, depending on the value of contract offered in each submarket and the probability of obtaining the job there. The numbers of firms and workers entering a submarket determine how tight the submarket is (*tightness of submarket*), and a free-entry condition adjusts the tightness so that firms are indifferent among entering any submarkets (*zero-profit condition*).

Once a firm and a worker form a match and sign a contract, they perform a series of projects: one in each period. Each project takes only two outcomes: success or failure, and the probability of success depends stochastically on the worker's unobservable effort. The series of projects continues as long as the project keeps succeeding, but if a project fails at any time, the match separates and the worker needs to search for a new job.

Firms can commit to a long-term contract, while employed workers can quit their jobs in any period if they find new and better contract through on-the-job search. A long-term contract can depend only on the tenure length of the worker on the current contract. This implies that a firm cannot respond to the worker's outside offer and that the period wage cannot depend on the current outcome of the project. Therefore, firms need to design a long-term contract so that it can induce the worker to stay on the contract as well as to make an appropriate level of effort.

I find that the optimal long-term contract exhibits an increasing wage-tenure profile. It is so for two reasons: 1) it can induce the workers more likely to stay in their current contracts, and 2) it can induces the workers to make efforts when the current up-front wages cannot. The former is a standard result in the labor search literature with on-the-job search (e.g., Burdett and Coles (2003) and Shi (2008)). Given the worker mobility, a firm has an incentive to backload wages to entice the worker to stay; as wages rise with tenure, it is more difficult for the worker to find a better offer elsewhere, and so the worker's quit rate falls. On the other hand, the latter is new in the literature. Since the current wage does not depend on outcomes of the current project, it cannot give the worker incentives to make effort for the current period project. The firm, however, can promise to give higher wage in the next period, and thus higher continuation value, if the current project succeeds. Hence, the worker's motivation for staying employed to receive higher continuation value in the firm and obtain better outside offer can give incentive for the worker to make effort for the current project. Overall, an increasing wage profile is less costly to the firm than a constant profile that promises the same value to the worker.

I find that the optimal incentive-compatible effort has also an increasing profile. This is due to a combined effect between 1) the workers' fear of loosing their jobs and 2) their incentive to obtain better outside offers. Given an increasing wage-tenure profile, a worker's value of the current contract increases, and thus cost of loosing the job after a failure increases over time. Also, given theincreasing value of the current contract, the worker's value of optimal search also increases. They both increase the expected benefit of effort making over time, and the worker keeps making more and more effort over the tenure.

I then show that an equilibrium with these feature indeed exists. The equilibrium inherits the "block recursivity" developed by Shi (2008) and Menzio and Shi (2008); that is, individuals' optimal decisions and optimal contracts are independent of the distribution of workers.

This paper makes contribution in such literature as dynamic contracting, contracting in general equilibrium, and equilibrium labor search. Literature on dynamic contracting is enormous, but this paper is especially related to Rogerson (1985), Holmstrom and Milgorm (1987), Spear and Srivastava (1987), and Sannikov (2008). These papers focus on contracting between a single firm and a single worker, and the outside environment is kept constant over time. Since contracting agents are not allowed to look for another contracts in the market, the theory does not address the issue of mobility which is the key aspect of this paper.

Key papers in the literature on contracting in general equilibrium theory are Prescott and Townsend (1984), and Phelan (1995). They consider contracting problems in which agents can move across contracts in competitive equilibrium frameworks. Common feature of these models is that contracting agents are perfectly mobile in the markets. On the other hand, the present paper focuses on frictional mobility in a market and shows how it affects the resulting optimal contract.

Closely related papers in the equilibrium labor search literature are Burdett and Coles (2003), Menzio and Shi (2008), and Shi (2008). They consider labor market matching models and shed new lights on interactions between employment contracting problems and frictional mobility via on-the-job search in labor markets. However, they fail to incorporate issues of asymmetric information, and incentive problems within the contractual relationship are abstracted from these analysis.

There are several recent papers to integrate these areas of research to obtain novel insights. Shimer and Wright (2004) study a static employment contracting problem with search frictions and bilateral asymmetric information. Manoli and Sannikov (2005) studies a dynamic employment contracts in an environment with competition between firms and with informational frictions. Board (2007) characterizes the optimal self-enforcing contracts in a large anonymous labour market which allows for on-the-job search by workers. They all try to examine further implications of contract theory by placing it in more realistic contexts, and the current paper shares this spirit.

This paper, however, presents some distinct features. First, in my model, a worker's outside environment has definite interaction with his behavior inside the firm, and it evolves endogenously via worker mobility as an optimal outcome. Then, the directed search framework enable me to characterize the equilibrium contract without referring the distribution of workers in the market and to prove the existence of an equilibrium with such a contract. Because of these features, I can obtain very clear predictions on the optimal contract.

2 A Model of Labor Market with Search Friction and Moral Hazard

2.1 Physical Environment

I consider a labor market with a continuum of ex-ante homogeneous, infinitely lived workers with measure 1 and a continuum of ex-ante homogeneous firms whose measure is determined by competitive entry. Time is discrete and continues forever. Each worker has a utility function $u(w)$ where w is income in a period. I assume $u : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, weakly concave, and the first derivative is bounded from both below and above, i.e., $u'(w) \in [\underline{u}', \bar{u'}]$ for all w. When employed, each worker exerts costly effort, e, for the project of the firm in each period where $e \in \mathbb{R}_+$. I assume cost function $c : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, weakly convex. Each worker maximizes the expected sum of lifetime utilities minus costs of effort discounted at the rate $\beta \in (0, 1)$, that is, $\sum_{t=0}^{\infty} \beta^t(u(w_t) - c(e_t))$.

Firm's project outcome takes only two values: $\{0, y\}$. When outcome is y the worker's performance is a "success," and when 0 it is a "failure." The probability of success depends on an effort level by the worker and is given by $r(e)$ where $r : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable, strictly increasing and weakly concave in e. I also assume that $r'(0) < \infty$ and $\lim_{e\to\infty} r(e) = 1$. Each firm maximizes the expected sum of profits discounted at the rate β.

There is a continuum of labor markets indexed by x, where x denotes the value of contract offered to a worker in that submarket, i.e., whenever a vacant firm meets a worker in that submarket it has to offer him an employment contract that gives him the lifetime utility x. I assume that $x \in X \subseteq \mathbb{R}_+$. Let G a cumulative distribution of employed workers over X and u a fraction of unemployed workers.The ratio of vacant firms to searching workers in submarket x is denoted by $\theta(x)$ and refered to as the tightness of submarket x.

In each period, there are two stages: search and matching, and production. In the search and matching stage, firms post a vacancy at a flow cost $k > 0$ and announce long-term contract to recruit a worker. Firms can direct the workers' search by offering in different submarket from each other. Each worker, independent of the employment status, receives the opportunity of searching for a job with a probability $\lambda \in [0, 1]$. If a worker has the opportunity of searching the labor market and choose to visit submarket x, he meets a vacant firm with probability $p(\theta(x))$, where $p : \mathbb{R}_+ \to [0, 1]$ is twice continuously differentiable, strictly increasing and strictly concave and such that $p(0) = 0, p'(0) < \infty$. On the other hand, a vacant firm finds a worker with probability $q(\theta(x))$, where $q : \mathbb{R}_+ \to [0, 1]$ is twice continuously differentiable, strictly decreasing and strictly convex and such that $\theta^{-1}p(\theta) = q(\theta), q(0) = 1$, and $p(q^{-1}(\cdot))$ is concave.

If an employed worker receives the opportunity of searching and matches with a firm, and if he accept the offer, he must leave his previous employment position before entering the production stage with a new firm. If he rejects the offer, he returns to his previous employment position and enters the production stage.

In the production stage, each unemployed worker produces at home and consumes b unit of output. Each employed worker receives the current period wage before the production takes place. Therefore, the wage cannot depend on the project outcome as in standard moral hazard literature. Then the worker chooses how much effort to make, and the project outcome is realized depending on the worker's effort level.

At the end of the period after the project outcome is realized, if project fails, the match is destroyed and the worker looses his employment position. If the project succeeds, the match stays together and the worker keeps his employment position in the next period. Projects continue forever as long as they are succeeding. There is no exogenous separation of the match and failure is the only reason of separation.

2.2 Contractual Environment

I assume firms can commit to long-term contracts but workers can quit a job at any period. In particular, when an employed worker receives an opportunity to search for an outside contract, the firm cannot respond to the employee's outside offers. Firms must design contracts that specifies worker's wage and the recommended effort level in each period. Employment contracts can specify only the worker's wage as afunction of his tenure on the job conditional on that the worker stays with the firm and -possibly-of the outcome of a randomization device. As mentioned, wage in each period does not depend on the realization of the project outcome. Workers receive up-front payments at the beginning of each production stage. Since the worker's effort level is unobservable to the firm, the recommended efforts need to be incentive compatible to the worker.

2.3 Equilibrium Conditions

In the following, I describe the equilibrium conditions for this economy. I will set up the optimal contracting problem as a recursive problem following the approach taken by Spear and Srivastava (1987) and much of the recent dynamic game literature.

First, I will explain a market tightness from firms free entry condition. Second, given the market tightness function, I will illustrate worker's problems in terms of optimal search and optimal effort making when employed and present the value of optimal search for employed workers. Third, I will give the value of unemployed worker. Finally, I will present the firm's optimal contracting problem as a recursive problem.

2.3.1 Market Tightness and Free Entry Condition

During the search stage, firms choose how many vacancies to create and where to locate them. Let $J(x)$ denote a firm's value of meeting a worker in submarket x. Then, the firm's expected benefit of creating a vacancy in submarket x is given by $q(\theta(x))J(x)$, the product of the probability and the value of meeting a worker in the submarket. Given the market tightness function $\theta(x)$, if the cost k of creating a vacancy is strictly greater than the expected benefit, then firms do not create any vacancies in submarket x. If k is strictly smaller than the expected benefit, then firms create infinitely many vacancy in x. When they are equal, then firms expected profits are zero and independent from the number of vacancies they create in submarket x. Therefore, in any x, $\theta(x)$ is consistent with firms' profit maximization if

$$q(\theta(x))J(x) - k \leq 0, \tag{1}$$

and $\theta(x) \geq 0$, with complementary slackness. The complementary slackness condition ensures that if the expected benefit is less than the cost of creating vacancy, then no firms create vacancy in submarket x; that is $\theta(x) = 0$. Also, free entry and exit condition drives the maximized expected profit to zero.

2.3.2 Worker's Problems

Optimal Search of the Worker

Suppose a worker's current value of employment position is V. If he receives the opportunity to search for a new contract and visits submarket x, he find an employer with probability $p(\theta(x))$

and yields the additional value of $x - V$. The worker chooses which submarket to visit to maximize the expected value of search. I will denote the worker's value of search as

$$D(V) = \max_{x \in \mathbb{R}} p(\theta(x))(x - V), \tag{2}$$

given the current value of employment position V. I denote with $m(V)$ the worker's optimal search policy of this problem, and denote the composite function $\hat{p}(V) = p(\theta(m(V)))$.

Optimal Effort Choice of the Worker

Consider a worker in the production stage. If the project succeeds, the current contract will provide a continuation value to the worker, W, and it will be the worker's reservation value from the current contract. In the next search stage, if he receives the opportunity to search with probability λ, he will obtain the expected benefit of $W + D(W)$ through the optimal search explained above. If he does not receive the opportunity to search, he keeps the reservation value W. On the other hand, if the project fails, the worker will loose the job in the next period and spend as an unemployed receiving the value of unemployed U as explained below. Hence, the net expected benefit of success with the effort level e is

$$r(e)(\lambda(W + D(W)) + (1 - \lambda)W) + (1 - r(e))U = r(e)(W + \lambda D(W)) + (1 - r(e))U.$$

The cost of effort level e is $c(e)$. Therefore, given the continuation value W, the worker will choose his effort level to solve:

$$\max_{e \in \mathbb{R}} \Big(- c(e) + \beta\big(r(e)(W + \lambda D(W)) + (1 - r(e))U\big)\Big).$$

Given the structure of the contract, current period wage is independent of the project outcome and does not affect the worker's optimal effort choice. Note that the expected benefit from the effort is given in the next period and the worker discount its value at β.

Worker's Value of Unemployment

Finally, consider an unemployed worker at the beginning of the production stage. The worker obtains $u(b)$ from the home production. Let U be the value of unemployed. If he receives opportunity to search for a job, he will obtain the expected benefit of $U + D(U)$ through the optimal search. If he does not receive the opportunity to search, he will stay unemployed and receive U. Therefore, the value of unemployed is expressed recursively as

$$U = u(b) + \beta\big(U + \lambda D(U)\big). \tag{3}$$

2.3.3 The Firm's Optimal Contracting Problem

Consider a firm that promises to provide a continuation value V in this period. Let $J(V)$ denote the current value of contract for the firm. The firms chooses: i) w, how much wage to pay in this period, ii) e, how much effort to induce, and iii) W, how much continuation value to provide to the worker in the next period conditional on the survival of the relationship. I also allow for randomization over these choices, that is the firm offer two sets of contract and a probability distribution, $\{\pi_i\}_{i=1,2}$, over these subcontracts. Denote by $\xi = (\{w_i, e_i, W_i, \pi_i\}_{i=1,2})$ the contract offered by the firm at the beginning of a period.

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The firm's optimal contracting problem is given by

$$J(V) = \max_{\xi} \sum_{i=1,2} \pi_i \left\{ r(e_i)y - w_i + \beta r(e_i)(1 - \lambda \hat{p}(W_i))J(W_i) \right\} \tag{4}$$

subject to

$$V = \sum_{i=1,2} \pi_i \left\{ u(w_i) - c(e_i) + \beta[r(e_i)(W_i + \lambda D(W_i)) + (1 - r(e_i))U] \right\},$$

$$e_i \in \arg\max_{e \in \mathbb{R}} \Big(-c(e) + \beta\big(r(e)(W + \lambda D(W)) + (1 - r(e))U\big)\Big), \quad \text{for } i = 1, 2,$$

and

$$\pi_1 + \pi_2 = 1.$$

By inducing an effort level e and paying w, the firm's expected current period payoff is $r(e)y - w$. In the next period, by offering continuation value W, the current worker will stay in the current firm with probability $r(e)(1 - \lambda\hat{p}(W))$: product between a probability the project succeeds and a probability the worker will not leave for the outside option, and the firm will enjoy the value $J(W)$ of remaining contract.

To design the contract, the firm faces three constraints. The first is the promise-keeping or consistency constraint, that is, the contract has to provide the worker with the promised value V. Since offer of the contract is evaluated ex-ante, only the expected value over the values of subcontracts must provide the promised continuation value; either subcontract may fail to provide the promised value. Then, the effort that the firm wants to induce must be incentive compatible, that is, given the contract offered the worker voluntarily choose to make that level of effort. Since realization of subcontract occurs before the worker chooses his effort level, both subcontracts the firm prepares need to meet the incentive compatibility constraint. Finally, when the firm randomizes the contracts, the probabilities assigned to each subcontract must sum to one. I denote with $\xi(V)$ the optimal policy functions given V associated with this contracting problem.

2.4 Block (Distribution-free) Recursive Equilibrium

Definition: A Recursive Equilibrium *is a set of functions* $\{J^*, \theta^*, D^*, m^*, U^*, \xi^*\}$ *and a distribution of workers* (G^*, u^*) *such that*

1. θ^* *satisfies condition (1),*
2. D^* *and* m^* *satisfies the functional equation (2),*
3. U^* *satisfies the functional equation (3),*
4. J^* *and* ξ^* *satisfies the functional equation (4), and*
5. (G^*, u^*) *is consistent with* m^* *and* ξ^**.*

Definition: A Block (Distribution-free) Recursive Equilibrium *is a recursive equilibrium such that functions* $\{J^*, \theta^*, D^*, m^*, U^*, \xi^*\}$ *are independent from the distribution of workers* (G^*, u^*)*.*

3 Equilibrium for the Optimal Long-Term Contract

In this section, we will first describe general properties of equilibrium and characterize the optimal long-term contract. Then I will show the existence of equilibrium in this environment. Proofs of each lemma is almost identical to that in Menzio and Shi (2008) or requires only minor modifications for considering worker's effort choice. I collect all the proofs for those requiring extra steps or modifications in the appendix.

We start with specifying a set of functions with certain properties. Then, we take an arbitrary function as a firm's value function and characterize the equilibrium objects and the optimal contract given the properties of functions in the set. With all these results, we construct an operator defined over the set of functions and show that any fixed point of the operator is a Block Recursive Equilibrium.

3.1 General Properties

We define the set $\mathcal{J}(X)$ of functions $J : X \to \mathbb{R}$ such that:

(i) $J(V)$ is strictly decreasing and Lipschitz continuous with respect to V,

(ii) $J(V)$ is bounded both from below and above, and

(iii) $J(V)$ is concave.

We can show that the set $\mathcal{J}(X)$ is non-empty, bounded, closed, and convex subset of the space of bounded, continuous functions on X with the sup norm.

3.1.1 Market Tightness and Free Entry Condition

First, we take an arbitrary firm's value function $J \in \mathcal{J}(X)$ and solve the equilibrium condition (1) with respect to the market tightness function θ. We get

$$\theta(x) = \begin{cases} q^{-1}(k/J(x)) & \text{if } \ J(x) \geq k \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Because q is a probability, q^{-1} is defined only on $[0, 1]$, that is $J(x) \geq k$. Since $J(x)$ is strictly decreasing, there exists a unique $\hat{x} \in \mathbb{R}$ such that $J(x) > \hat{x}$ for all $x < \hat{x}$ and $J(x) < \hat{x}$ for all $x > \hat{x}$. That is, offering any contract with the value more than $\hat{x}$ provide negative profits even if the firm can hire a worker with probability one. Therefore, no firm will enter submarkets with $x > \hat{x}$, and the market tightness takes nonnegative value only if $x \leq \hat{x}$. We call this threshold value an *effective bound* given J. Since J is bounded from above, θ is also bounded from above. Now we have the following properties of market tightness functions.

Lemma 3.1. *If $x < \hat{x}$, the market tightness function $\theta(x)$ is strictly positive and strictly decreasing, and Lipschitz continuous.$\theta(x) = 0$ for all $x \geq \hat{x}$*

3.1.2 Worker's Problem

Optimal Search of the Worker

In this section, we insert the market tightness function, $\theta(x)$, obtained from an arbitrary firm's value function $J \in \mathcal{J}(X)$ into the worker's search problem:

$$D(V) = \max_{x \geq V} p(\theta(x))(x - V).$$

Now, we have the following series of properties which enable to compute the optimal contracts.

Lemma 3.2. *For all $V \in X$, the worker's objective function $f(x;V) = p(\theta(x))(x - V)$ is strictly concave with respect to x.*

The concavity of the problem implies that, given a continuation value of the current contract, the worker will find unique submarket that he optimally visits.

Lemma 3.3. *The worker's optimal search strategy $m(V) \in \arg\max p(\theta(x))(x - V)$ is unique, weakly increasing and Lipshitz continuous.*

The lemma also implies that a worker searches in a submarket which offers a higher value the higher your continuation value from the current contract. This can be seen as a version of the single crossing property.

Lemma 3.4. *If $V < \hat{x}$, the worker's value of searching $D(V)$ is strictly positive and weakly decreasing and Lipschitz continuous. If $V \geq \hat{x}$, $D(V) = 0$.*

As long as a worker's current continuation value is less than the effective bound $\hat{x}$, there are submarkets that offer higher values and they provide positive expected values. However, once his continuation value hits the bound, there are no firms offering better offer and the market tightness becomes zero, $D(V) = 0$ since $p(0) = 0$.

Lemma 3.5. *$\hat{p}(V)$ is weakly decreasing and Lipschitz continuous.Moreover $\hat{p}(\hat{x}) = 0$*

Remember that the composite function $\hat{p}(V)$ is a probability that a worker with a continuation value V from the current contract meets a firm in the optimally targeted submarket. Since a worker with higher V applies to a submarket with higher tightness, he is less likely to meet a firm in an optimally targeted submarket.

Optimal Effort Choice of the Worker

We now turn to a worker's optimal effort choice problem:

$$\max_{e \in \mathbb{R}} \Big(- c(e) + \beta\big(r(e)(W + \lambda D(W)) + (1 - r(e))U\big)\Big).$$

Under the assumption on $c(\cdot)$ and $r(\cdot)$, it is a concave problem with respect to e, and we applies the first-order approach to characterize the optimal effort.

Lemma 3.6. *Given a continuation value W, there is a unique level of optimal effort. The worker's optimal effort function is increasing in the continuation value of contract, W.*

Proof: First, for a given continuation value W, the worker's optimal effort is implicitly and given by

$$-c'(e) + \beta r'(e)(W + \lambda D(W) - U) = 0.$$

Define $\Omega(W) = W + \lambda D(W) - U$. Then we can write $\frac{c'(e)}{\beta r'(e)} = \Omega(W)$. Under the assumptions ($c(\cdot)$: convex, continuous, $r(\cdot)$: concave continuous, and $r' > 0$ everywhere), $\frac{c'}{\beta r'}$ is continuous and monotone increasing, and thus invertible. Therefore, there exists a unique level of e to satisfy the equality.

Moreover the inverse function is also continuous. Hence, we write $e(W) = g(\Omega(W))$ where g is the inverse function of $\frac{c'}{\beta r'}$. Differentiating the composite function,

$$\begin{aligned}
\frac{\partial e(W)}{\partial W} &= g'(\cdot)\Omega'(W) \\
&= \left(\left(\frac{c'}{\beta r'}\right)^{-1}\right)' (1 - \lambda \hat{p}(W)) \\
&= \left(\frac{c''r' - c'r''}{\beta r'^2}\right)^{-1} (1 - \lambda \hat{p}(W)) \\
&= \frac{\beta r'^2}{c''r' - c'r''}(1 - \lambda \hat{p}(W)),
\end{aligned}$$

where the first equality is by the chain rule and the third is by the inverse function theorem. Since the denominator of the right hand side is strictly positive by assumption, and thus the derivative is positive, the worker's optimal effort is increasing in W. □

3.2 Characterization of the Optimal Contracts

We now characterize the optimal contract. Given all the previous results, a firms problem is given by the following recursive form:

$$J(V) = \max_{\xi} \sum_{i=1,2} \pi_i \left\{ r(e_i)y - w_i + \beta r(e_i)(1 - \lambda \hat{p}(W_i))J(W_i) \right\}$$

subject to

$$\begin{aligned}\xi \in \Xi = \Big\{ & \{w_i, e_i, W_i, \pi_i\}_{i=1,2}: \quad W_i \in [0, \hat{x}] \ \text{ for } i = 1,2 \\ & V = \sum_{i=1,2} \pi_i \left\{ u(w_i) - c(e_i) + \beta[r(e_i)(W_i + \lambda D(W_i)) + (1 - r(e_i))U] \right\} \\ & - c'(e_i) + \beta r'(e_i)(W_i + \lambda D(W_i) - U) = 0, \quad \text{for } i = 1,2 \\ & \pi_1 + \pi_2 = 1, \quad \pi_i \in [0,1] \ \text{ for } i = 1,2 \Big\}\end{aligned}$$

First, we consider only the promise-keeping constraint and the incentive compatibility constraint to obtain the following result.

Proposition 3.1. *Under the optimal contract, the current period wage is independent of the realization of lottery.*

Proof. Let $\eta(V)$ be the Lagrange multiplier for the promise-keeping constraint. The first order condition for w_i implies

$$\eta(V) = \frac{1}{u'(w_i)} \quad \text{for} \quad i = 1, 2.$$

This implies that $w_1 = w_2$. Therefore, the current wage, w, does not depend on the realization of the lottery. □

Given the above proposition, the firm's problem can be greatly simplified. It essentially makes it as a one-variable, unconstrained problem, at least for our characterization purpose.

Lemma 3.7. (Reduction of the Problem) *Let $\gamma = (\{W_i, \pi_i\}_{i=1,2})$. The optimal contracting problem can be reduced to the following optimization problem with respect to γ.*

$$\begin{aligned}J(V) = \max_{\gamma} \Bigg\{ & - u^{-1}\left(V - \beta U - \sum_i \pi_i \Big[- c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda D(W_i) - U] \Big] \right) \\ & + \sum_i \pi_i r(e(W_i)) \Big[y + \beta(1 - \lambda \hat{p}(W_i)) J(W_i) \Big] \Bigg\}\end{aligned} \tag{6}$$

where

$$\gamma \in \Gamma = \Big\{ \{W_i, \pi_i\}_{i=1,2}: \ W_i \in [0, \hat{x}], \quad \pi_i \in [0,1] \ \text{ for } i = 1,2 \ \text{ and } \pi_1 + \pi_2 = 1 \Big\}$$

Proof. Substituting the optimal effort function into the objective function and the first constraint eliminate the incentive compatibility constraint. Then, using the result that $w_1 = w_2$ makes the promise-keeping constraint as

$$V = u(w) + \beta U + \sum_i \pi_i \Big[-c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda D(W_i) - U] \Big].$$

Since u is strictly concave and thus invertible, this constraint can be solved for w. Substituting resulting expression for w into the objective function gives the desired form of unconstrained optimization problem. □

3.2.1 The Optimal Wage and Continuation Value Processes

We are interested in a shape of long-term wage-tenure profile in this environment. The following result seems to be a standard result of backloaded wage profile in the dynamic on-the-job search literature, but gives and additional insight into the mechanism. Moreover, from the dynamic moral hazard perspective, this kind of characterization seems to be a novel result.

Proposition 3.2. *Under the optimal contract, wages are nondecreasing over the tenure if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$ they are strictly increasing, and they stay constant once the continuation value hits the effective bound $\hat{x}$.*

Proof: First, ignoring the effective bound constraint, the first order condition of the reduced form of the problem (6) with respect to W_i is

$$\begin{aligned}
&- \frac{1}{u'(w)} \cdot (-\pi_i \beta r(e(W_i))(1 - \lambda \hat{p}(W_i))) \\
&+ \pi_i r'(e(W_i))[y + \beta(1 - \lambda \hat{p}(W_i))J(W_i)]g'(\cdot)(1 - \lambda \hat{p}(W_i)) \\
&+ \pi_i r(e(W_i))\beta\big[(1 - \lambda \hat{p}(W_i))J'(W_i) - \lambda J(W_i)\hat{p}'(W_i)\big] = 0
\end{aligned}$$

where we use that $\frac{\partial \Omega(W)}{\partial W} = 1 - \lambda \hat{p}(W)$ and $e'(W) = g'(\cdot)\Omega'(W)$. Then, dividing through by $\beta, \pi_i, r(e), (1 - \lambda \hat{p}(W_i))$ gives

$$-\frac{1}{u'(w)} + \frac{r'(e(W_i))}{r(e(W_i))} \left(\frac{y}{\beta} + (1 - \lambda \hat{p}(W_i)) \right) g'(\cdot) + J'(W_i) - \frac{\lambda J(W_i)\hat{p}'(W_i)}{1 - \lambda \hat{p}(W_i)} = 0.$$

Now, from the first order condition of the constrained problem, we have $\eta(V) = \frac{1}{u'(w(V))}$. The theorem of Lagrangian multiplier implies that $J'(V) = -\eta(V)$, so we have

$$J'(V) = -\frac{1}{u'(w(V))}. \tag{7}$$

Shifting one period forward gives $J'(W_i) = -\frac{1}{u'(w(W_i))}$ where $w(W_i)$ is the wage in the next period when the current lottery realization is i.

Substituting it into the previous equation and rearranging the terms gives

$$\frac{1}{u'(w(W_i))} - \frac{1}{u'(w)} = -\frac{\lambda J(W_i)\hat{p}'(W_i)}{1 - \lambda\hat{p}(W_i)} + \frac{r'(e(W_i))}{r(e(W_i))}\left(\frac{y}{\beta} + (1 - \lambda\hat{p}(W_i))\right)g'(\cdot). \tag{8}$$

The right hand side is positive since $\hat{p}'(W_i)$ is non-positive. Therefore, we have

$$\frac{1}{u'(w(W_i))} - \frac{1}{u'(w)} > 0. \tag{9}$$

Since u is concave function, this implies $w(W_i) > w$ for $i = 1, 2$. Hence, the next period wage is higher than the current wage.

However, given the effective bound constraint, ever increasing wage profile is not optimal; after some point, the firms is paying more than the project produce. Once a worker's value of contract hits the effective bound $\hat{x}$, in equilibrium, there is no other firms offering higher value and $\hat{p}(\hat{x}) = 0$, so the worker has no incentive to search on-the-job. The firm keeps the value and pays the wage

$$\hat{w} = -u^{-1}\big(\hat{x} - \beta U + c(e(\hat{x})) - \beta r(e(\hat{x}))(\hat{x} - U)\big).$$

□

The equation (8) in the proof gives explicit mechanics of backloaded wage-tenure profile; the first term considers the worker's mobility whereas the second term considers the moral hazard problem. Given the worker's mobility, a firm has an incentive to backload wages to entice the worker to stay; as wages rise with tenure, it is more difficult for the worker to find a better offer elsewhere, and so the worker's quit rate falls (see Shi 2008). On the other hand, the second term is new in the literature. Since the current wage does not depend on outcome of the current project, the current period wage cannot give a worker incentives to make effort for the current period project. The firm, however, gives incentives to promise higher continuation value in the next period. Overall, a rising wage profile is less costly to the firm than a constant profile that promises the same value to the worker.

Corollary 3.1. *Under the optimal contract, the next period promised continuation values are nondecreasing over the tenure if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$, they are strictly increasing, and they stay constant once they hit the effective bound $\hat{x}$.*

Proof: If $V < \hat{x}$, with (7), the inequality (9) implies $J'(W_i) - J'(V) < 0$. Then, concavity of J implies $W_i > V$ for $i = 1, 2$. If $V \geq \hat{x}$, optimal offer stays constant at $\hat{x}$. □

3.2.2 The Optimal Incentive Compatible Effort Process

Proposition 3.3. *Under the optimal contract, the worker's optimal efforts are nondecreasing if the worker stays on the contract, independent of the realization of lottery. Especially, if $V < \hat{x}$, they are strictly increasing, and they stay constant once the continuation value hits the effective bound $\hat{x}$.*

Proof: As shown in lemma 3.6, the optimal effort function is increasing in the continuation value of the contract. Since the continuation value of the contract is higher than the current value of the contract (Corollary (3.1)), the worker will make higher effort next period than the current period effort if stays on the contract. □

This is due to a combined effect between 1) a worker's fear of loosing the job and 2) his incentive to obtain the better outside offer. Given rising wages, a worker's value of the current contract increases over time and cost of loosing the job is increasing. Also, given increasing value of the current contract, worker's value of optimal search increases. Both of these factors give the worker incentives to make higher and higher efforts. These, in turn, sustain the increasing wage-tenure profile offered by the firm. In some sense, moral hazard problems are ameliorated through the market forces.

3.3 Existence of a Block Recursive Equilibrium

So far, we have examined properties of equilibrium objects and characterized the optimal contracts given an arbitrary firm's value function $J \in \mathcal{J}(X)$. In this section, we establish that such an equilibrium, especially a Block Recursive Equilibrium exists.

Proposition 3.4. *A Block Recursive Equilibrium exists.*

Detailed proof is given in the appendix. I will only outline the argument in the proof. First, by inserting all the previous equilibrium objects, given an arbitrary value function $J \in \mathcal{J}(X)$, into a firm's optimal contracting problem, we construct the following operator that maps from $\mathcal{J}(X)$ to some space of functions:

$$(TJ)(V) = \max_{\xi \in \Xi} \sum_{i=1,2} \pi_i \left\{ r(e_i)y - w_i + \beta r(e_i)(1 - \lambda \hat{p}(W_i))J(W_i) \right\}.$$

We show that T is a self-map, that is, T maps from $\mathcal{J}(X)$ to itself (Lemma 5.8). Then, we show that T is a continuous map (Lemma 5.9). These suffice to show that the operator T satisfies the assumptions of Schauder Fixed Point Theorem (Stokey and Lucas with Prescott, 1989, Theorem 17.4); that is, (i) T is continuous, (ii) the family of functions $T(\mathcal{J})$ is equicontinuous, and (iii) T maps the set $\mathcal{J}(X)$ into itself. Therefore, there exists a firm's value function $J^*(V) \in \mathcal{J}(X)$ such that $TJ^* = J^*$. Denote with $\{\theta^*, D^*, m^*, U^*, e^*, \xi^*\}$ the respective functions associated to J^*. By construction, the functions $\{J^*, \theta^*, D^*, m^*, U^*, e^*, \xi^*\}$ satisfy conditions in the definition of a Recursive Equilibrium and do not depend on the distribution of workers. Therefore, they constitute a Block Recursive Equilibrium.

4 Conclusion

In this paper, I developed a search theoretic general equilibrium model of an employment contracting with repeated moral hazard. The optimal contract exhibits an increasing wage-tenure profile as well as an increasing incentive-compatible effort profile. Moreover, I showed that an equilibrium exists and is independent of the distribution of workers.

I am currently analyzing how the presence of moral hazard problem affects the distribution of workers. Since the equilibrium is independent from the distribution, it is possible to analyze the evolution of the distribution out of the steady state. I am also examining impacts of some policies, such as a minimum wage law or change in unemployment benefit, on the optimal contract and on the distribution of workers. While, in this paper, I focused exclusively on positive aspects of the benchmark model, it may also be interesting to investigate efficiency or normative properties of the equilibrium.

There are several potential extensions and future research areas. First, it is important to investigate workers' saving behavior since, in my model, there are always risks of loosing their job. Allowing the workers to access to a credit market is an important extension. Other more challenging extensions include 1) introducing heterogeneous firms (sectors) to examine workers' inter-sectorial mobility, 2) allowing firms to choose either fire or retain the worker after a project fails, 3) introducing costly search on workers' side, and 4) considering delayed effects of worker's effort on the project outcome.

Lastly, as a companion part of my ongoing research project, I am studying a case of repeated adverse selection in which workers repeatedly receive unobservable productivity shocks or consumers repeatedly receive unobservable endowment shocks as in Phelan (1995). Incorporating other cases of asymmetric information seems to be an important research area.

5 Appendix

5.1 Other Properties of Equilibrium

Note that, in the text, all the objects in the characterizations depend on a specific $J \in \mathcal{J}(X)$. To show the existence of a fixed point of the operator $TJ(V)$, we need to show that they are continuous with respect to $J \in \mathcal{J}(X)$. The following series of lemmas imply the continuity of them. Almost all are shown in Menzio and Shi (2008) and proofs are omitted.

5.1.1 Free Entry Condition and Market Tightness

Lemma 5.1. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\theta_j(x)$ be the market tightness function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\theta_m - \theta_n|| < \varepsilon_\theta \rho$.*

5.1.2 Worker's Search Problem

Lemma 5.2. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $D_j(V)$ be the worker's value of searching implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||D_m - D_n|| < \varepsilon_D \rho$.*

Lemma 5.3. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $m_j(V)$ be the optimal search strategy implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||m_m - m_n|| < \varepsilon_m \rho$.*

Lemma 5.4. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\hat{p}_j(V) = p(\theta_j(m_j(V)))$ be the conposite function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\hat{p}_m - \hat{p}_n|| < \varepsilon_p \rho$.*

5.1.3 Worker's Value of Unemployment

Lemma 5.5. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let U_j be the worker's unemployment value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||U_m - U_n|| < \varepsilon_U \rho$.*

5.1.4 Worker's Optimal Effort

Lemma 5.6. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\Omega_j(W) = W + \lambda D_j(W) - U_j$ be the worker's net continuation value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\Omega_m - \Omega_n|| < \varepsilon_\Omega \rho$.*

Proof of Lemma 5.6:

$$\begin{aligned}&|\Omega_m(W) - \Omega_n(W)|\\ =& |\lambda(D_m(W) - D_n(W)) - (U_m - U_n)|\\ \leq& |\lambda\varepsilon_D - \varepsilon_U|\rho.\end{aligned}$$

Lemma 5.7. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $e_j(W) = g(\Omega_i(W))$ be the worker's optimal effort function implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||e_m - e_n|| < \varepsilon_e\rho$.*

Proof of Lemma 5.7:

Let $\bar{g'} = |\sup g'(\cdot)|$. Given the assumptions about $c(\cdot)$ and $r(\cdot)$, $\bar{g'} < \infty$. Then,

$$\begin{aligned}|e_m(W) - e_n(W)| &= |g(\Omega_m(W)) - g(\Omega_n(W))|\\ &\leq \bar{g'}|\Omega_m(W) - \Omega_n(W)|\\ &\leq \bar{g'}\varepsilon_\Omega\rho.\end{aligned}$$

5.2 Omitted Lemmas and Proofs

These lemmas extend the results shown in Menzio and Shi (2008) to show the existence of a Block Recursive Equilibrium when there is a moral hazard problem. Let $\hat{J}(V)$ be a firm's updated value function by the operator T, i.e., $\hat{J}(V) = (TJ)(V)$.

Lemma 5.8. *The firm's value function $\hat{J}(V)$ belongs to the set $\mathcal{J}(X)$. That is,*

(i) $\hat{J}(V)$ is strictly decreasing, and Lipschitz continuous with respect to V.

(ii) $\hat{J}(V)$ is bounded both from below and above.

(iii) $\hat{J}(V)$ is concave.

Proof of Lemma 5.8: The operator is a self-mapping.

(i) From the characterization result, let F be the objective function of the reduced problem, i.e.,

$$F(V,\gamma) = \Bigg\{ - u^{-1}\left(V - \beta U - \sum_i \pi_i\Big[- c(e(W_i)) + \beta r(e(W_i))[W_i + \lambda D(W_i) - U]\Big]\right) + \sum_i \pi_i r(e(W_i))\Big[y + \beta(1 - \lambda\hat{p}(W_i)J(W_i)\Big]\Bigg\}$$

where γ is a contract. Let Γ be the set of feasible contracts as defined in the text. By the Inverse Function Theorem,

$$F'(V,\gamma) = -\frac{1}{u'(w)} \in \left[-\frac{1}{\underline{u}'}, -\frac{1}{\bar{u}'}\right]$$

Now, for any $V_a, V_b \in X$, such that $V_a \leq V_b$, we have

$$\begin{aligned}
|\hat{J}(V_b) - \hat{J}(V_b)| &\leq \max_{\gamma\in\Gamma} |F(V_b,\gamma) - F(V_a,\gamma)| \\
&= \max_{\gamma\in\Gamma} \left| \int_{V_a}^{V_b} F'(V,\gamma) dV \right| \\
&\leq \max_{\gamma\in\Gamma} \int_{V_a}^{V_b} \left| F'(V,\gamma) \right| dV \\
&\leq \frac{1}{\underline{u}'} |V_b - V_a|
\end{aligned}$$

Therefore $\hat{J}(V)$ is Lipschitz continuous in V. From this result, $\hat{J}(V)$ is absolutely continuous and thus almost everywhere differentiable (Folland, 1999). Moreover, at any point of differentiability, we have $\hat{J}'(V) = F'(V,\gamma(V))$ where $\gamma(V)$ is the optimal contract given V (Milgrom and Segal, 2002). Then,

$$\hat{J}(V_b) - \hat{J}(V_a) = \int_{V_a}^{V_b} F'(V,\gamma(V))dV \in \left[-\frac{1}{\underline{u}'}(V_b - V_a), -\frac{1}{\bar{u}'}(V_b - V_a)\right]$$

Hence, $\hat{J}(V)$ is strictly decreasing and the difference is bounded.

(ii) Next, we estimate the bounds of $\hat{J}(V)$. Let $\underline{w}$ be the lowest possible wage under the feasible contract. That is

$$\underline{w} = \min_{\gamma\in\Gamma} u^{-1} \left(V - \sum_{i=1,2} \pi_i(-c(e(W_i)) + \beta[r(e_i)(W_i + \lambda D(W_i)) + (1 - r(e_i))U]) \right)$$

Since u' is increasing function and the expected continuation value for the worker is bounded by $\bar{x} = \sup_{J\in\mathcal{J}} \hat{x}_J$, which is finite, we have $\underline{w} \geq u'(\underline{x} + c(\underline{e}) - \beta\bar{x})$. Using the fact that $\hat{J}(V)$ is strictly decreasing in V, we have

$$\begin{aligned}
\hat{J}(V) &< \hat{J}(\underline{x}) \\
&\leq r(\bar{e})y - u'(\underline{x} + c(\underline{e}) - \beta\bar{x}) + \beta\bar{J} \equiv \bar{J}.
\end{aligned}$$

Then, $\hat{J}(V) \leq \bar{J} = \frac{y - u'(\underline{x} + c(\underline{e}) - \beta\bar{x})}{1-\beta}$.

Similarly to the previous argument,

$$\begin{aligned}
\hat{J}(V) &> \hat{J}(\bar{x}) \\
&\geq r(\underline{e})y - u'(\bar{x} + c(\bar{e}) - \beta U) + \beta\underline{J} \equiv \underline{J}.
\end{aligned}$$

Then, $\hat{J}(V) \geq \underline{J} = \frac{r(\underline{e})y - u'(\bar{x}+c(\bar{e})-\beta U)}{1-\beta}$. Hence, $\hat{J}(V)$ is bounded both from below and above.

(iii) Concavity of $\hat{J}(V)$ can be shown with two-point convexification result developed by Menzio and Shi (2008) and omitted in this paper. □

Lemma 5.9. *Consider $J_m, J_n \in \mathcal{J}(X)$. Let $\hat{J}_j(W)$ be the firm's value implied by J_j for $j = m, n$. If $||J_m - J_n|| < \rho$, then $||\hat{J}_m - \hat{J}_n|| < \varepsilon_T \rho$.*

Continuity of the operator with respect to $J \in \mathcal{J}(X)$.

Proof of Lemma 5.9: Continuity of the operator.

Let F_j be the objective function of the firms optimal contracting problem implied by J_j: $F_j : \Gamma \times X \to \mathbb{R}$. Consider $J_m, J_n \in \mathcal{J}(X)$ such that $||J_m - J_n|| < \rho$. Take $V \in X$ such that $\hat{J}_m(V) - \hat{J}_n(V) > 0$. Let γ_j be the maximizer of F_j and $w_j(\gamma)$ be the wage function given by J_j. Then, we have

$$
\begin{aligned}
0 &\leq |\hat{J}_m(V) - \hat{J}_n(V)| \\
&= |F_m(\gamma_m, V) - F_n(\gamma_n, V)| \\
&\leq |F_m(\gamma_m, V) - F_n(\gamma_m, V)| \\
&\leq \Big| - w_m(\gamma_m) + \sum_i \pi_{i,m} r(e_m(W_{i,m}))\left[y + \beta(1 - \lambda \hat{p}_m(W_{i,m}))J_m(W_{i,m})\right] \\
&\quad + w_n(\gamma_m) - \sum_i \pi_{i,m} r(e_n(W_{i,m}))\left[y + \beta(1 - \lambda \hat{p}_n(W_{i,m}))J_n(W_{i,m})\right] \Big| \\
&\leq |w_m(\gamma_m) - w_n(\gamma_m)| \\
&\quad + \sum_i \pi_{i,m} \Big| r(e_m(W_{i,m}))\left[y + \beta(1 - \lambda \hat{p}_m(W_{i,m}))J_m(W_{i,m})\right] \\
&\qquad\qquad - r(e_n(W_{i,m}))\left[y + \beta(1 - \lambda \hat{p}_n(W_{i,m}))J_n(W_{i,m})\right] \Big|.
\end{aligned}
$$

We want to estimate a bound for $|\hat{J}_m(V) - \hat{J}_n(V)|$. We will consider a bound for each part of the last expression separately as follows.

1. $|w_m(\gamma_m) - w_n(\gamma_m)|$:

Since u is concave function, for any w_1 and w_2, $|w_1 - w_2|u' < |u(w_1) - u(w_2)|$. Also,

$$u(w_m(\gamma_m)) = V - \beta U_m - \sum_i \pi_{i,m}[-c(e_m(W_{i,m})) + \beta r(e_m(W_{i,m}))\Omega_m(W_{i,m})$$

$$u(w_n(\gamma_m)) = V - \beta U_n - \sum_i \pi_{i,m}[-c(e_n(W_{i,m})) + \beta r(e_n(W_{i,m}))\Omega_n(W_{i,m})$$

Then

$$
\begin{aligned}
&|u(w_m(\gamma_m)) - u(w_n(\gamma_m))| \\
\leq\ & \beta|U_m - U_n| + \sum_i \pi_{i,m}\Big\{|c(e_m(W_{i,m})) - c(e_m(W_{i,m}))| \\
&\qquad + \beta|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})|\Big\}
\end{aligned}
$$

Now, consider the last part: $|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})|$.

$$
\begin{aligned}
&|r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})| \\
\leq\ & |r(e_m(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))\Omega_m(W_{i,m}) - r(e_n(W_{i,m}))\Omega_n(W_{i,m})| \\
=\ & |r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|\bar{x} + r(e_n(W_{i,m}))|\Omega_m(W_{i,m}) - \Omega_n(W_{i,m})| \\
\leq\ & r'(\underline{e})|e_m(W_{i,m}) - e_n(W_{i,m})|\bar{x} + |\Omega_m(W_{i,m}) - \Omega_n(W_{i,m})| \\
\leq & (r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega)\rho
\end{aligned}
$$

I use the fact that $\Omega(\cdot)$ is bounded by $\bar{x}$. Collecting them together, we have

$$
\begin{aligned}
&|u(w_m(\gamma_m)) - u(w_n(\gamma_m))| \\
\leq\ & (\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega))\rho.
\end{aligned}
$$

Hence

$$
|w_m(\gamma_m) - w_n(\gamma_m)| \leq u'^{-1} \cdot (\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega))\rho.
$$

2.

$$
\begin{aligned}
\sum_i \pi_{i,m}\Big| & r(e_m(W_{i,m}))\left[y + \beta(1 - \lambda\hat{p}_m(W_{i,m}))J_m(W_{i,m})\right] \\
& - r(e_n(W_{i,m}))\left[y + \beta(1 - \lambda\hat{p}_n(W_{i,m}))J_n(W_{i,m})\right]\Big| :
\end{aligned}
$$

This expression can still be divided into subcomponents after expanding the brackets and collecting terms:

(ii) $|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})|$:

$$\begin{aligned}
&|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})| \\
\leq\ &|r(e_m(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))J_m(W_{i,m}) - r(e_n(W_{i,m}))J_n(W_{i,m})| \\
=\ &|r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|J_m(W_{i,m}) \\
&+ r(e_n(W_{i,m}))|J_m(W_{i,m}) - J_n(W_{i,m})| \\
\leq\ &r'(\underline{e})|e_m(W_{i,m}) - e_n(W_{i,m})|\bar{J} \\
&+ |J_m(W_{i,m}) - J_m(W_{i,m})| \\
\leq\ &(r'(\underline{e})\varepsilon_e \bar{J} + 1)\rho.
\end{aligned}$$

(iii) $|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})|$:

$$\begin{aligned}
&|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_m(W_{i,m})| \\
&+ \ |\hat{p}_n(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
=\ &|\hat{p}_m(W_{i,m}) - \hat{p}_n(W_{i,m})|J_m(W_{i,m}) \\
&+ \ \hat{p}_n(W_{i,m})|J_m(W_{i,m}) - J_n(W_{i,m})| \\
\leq &(\varepsilon_p \bar{J} + 1)\rho.
\end{aligned}$$

(iv) $|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})|$:

$$\begin{aligned}
&|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &|r(e_m(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m})| \\
&+ |r(e_n(W_{i,m}))\hat{p}_m(W_{i,m})J_m(W_{i,m}) - r(e_n(W_{i,m}))\hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
=\ &|r(e_m(W_{i,m})) - r(e_n(W_{i,m}))|\hat{p}_m(W_{i,m})J_m(W_{i,m}) \\
&+ r(e_n(W_{i,m}))|\hat{p}_m(W_{i,m})J_m(W_{i,m}) - \hat{p}_n(W_{i,m})J_n(W_{i,m})| \\
\leq\ &((r'(\underline{e})\varepsilon_e \bar{J} + 1)\bar{J} + (\varepsilon_p \bar{J} + 1))\rho
\end{aligned}$$

Finally, putting everything together gives that

$$
\begin{aligned}
&|\hat{J}_m(V) - \hat{J}_n(V)| \\
\leq &u'^{-1} \cdot \left(\beta\varepsilon_U + c'(\bar{e})\varepsilon_e + \beta(r'(\underline{e})\varepsilon_e\bar{x} + \varepsilon_\Omega)\right)\rho \\
&+ \left\{(y + \beta + \beta\lambda\bar{J})(r'(\underline{e})\varepsilon_e\bar{J} + 1) + r(\bar{e})(\varepsilon_p\bar{J} + 1)\right\}\rho \\
\equiv &\varepsilon_T\rho.
\end{aligned}
$$

□

Proof of Proposition 3.4: Existence of a Block Recursive Equilibrium.

Given Lemma 5.8 and 5.9, we can show that the operator T satisfies the assumptions of Schauder's Fixed Point Theorem (Stokey and Lucas with Prescott, 1989, Theorem 17.4); that is, (i) T is continuous, (ii) the family of functions $T(\mathcal{J})$ is equicontinuous, and (iii) T maps the set $\mathcal{J}(X)$ into itself. Therefore, there exists a firm's value function $J^*(V) \in \mathcal{J}(X)$ such that $TJ^* = J^*$. Denote with $\{\theta^*, D^*, m^*, U^*, e^*\}$ the respective functions associated to J^*. By construction,the functions $\{J^*, \theta^*, D^*, m^*, U^*, e^*\}$ satisfy conditions in the definition of a Recursive Equilibrium and do not depend on the distribution of workers. Therefore, they constitute a Block Recursive Equilibrium.

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