All Equilibrium Revenues in Buy Price Auctions*

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Abstract

This note considers second-price, sealed-bid auctions with a buy price. We use a simple two-bidder, two-type framework and examine how the introduction of buy prices affects the seller's equilibrium revenues. When we consider all equilibria, an equilibrium revenue can be zero in the auctions without a buy price. On the contrary, an equilibrium revenue is always positive in the auctions with a buy price.

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Key words: Auction, Buy price

1 Introduction

This note considers second-price, sealed-bid auctions with a buy price. We analyze a simple two-bidder, two-type framework and examine how the introduction of buy prices affects the seller's equilibrium revenues. To highlight the effects of buy prices, we contrast the case of auctions without a buy price and the case of auctions with a buy price. Especially, we focus on the maximum (or supremum) and the minimum (or infimum) of equilibrium revenues. We consider all equilibria.[1]

When we consider the auctions without a buy price, as Maskin and Riley (1985) pointed out, there are a lot of imperfect equilibria even in a two-bidder, two-type framework.[2] Of course, an equilibrium revenue is not unique. It can be zero, or greater than the one at the weakly dominant strategy equilibrium.

Similarly, we have a lot of equilibria in the auctions with a buy price.[3] The equilibrium revenues, however, are not zero. That is, a seller can always obtain a positive equilibrium revenue by introducing a buy price. On the contrary, the maximum of equilibrium revenues does not change.

Several papers consider all equilibria, not all equilibrium revenues.[4] Blume and Heidhues (2004) derived all equilibria in second-price, sealed-bid auctions where there are at least three bidders. Plum (1992) analyzed all equilibria in two-bidder auctions with various payment rules. In second-price, sealed-bid auctions, however, he restricted their attention

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[1] Many papers often limit their attention to equilibria where all bidders play weakly dominant strategies. However, restrictions of this kind are not supported by theoretical or empirical results.

[2] Milgrom (1981) indicated that there are imperfect equilibria in continuous type distributions.

[3] Budish and Takeyama (2001) restricted their attention to the symmetric equilibrium where both bidders' high-types bid a buy price and both bidders' low-types bid their own valuations.

[4] To our knowledge, there is little related work on all equilibrium revenues.

to undominated strategy equilibria. Bikhchandani and Riley (1991) and Lizzeri and Persico (2000) analyzed interdependent value models.

The rest of this note is organized as follows. Section 2 describes the model. Section 3 presents the main result. And Section 4 concludes.

2 The model

We consider second-price, sealed-bid auctions with a buy price $B \in [0, +\infty)$.[5] We assume that bidders' types are independently and identically distributed. Moreover, we assume that bidders value the item depending on their own types.

Let $N = \{1, 2\}$ be the set of bidders. The set of types is $T_i = \{v^L, v^H\}$, where $v^L < v^H$. We denote by p the probability that bidder i is v^L-type. We assume that $p > 0$.

In Internet auctions, bidders do not bid above the buy price B. We then assume that bidders cannot bid above a buy price B. That is, the set of actions is $A_i = [0, B]$. Bidder i's payoff function is $u_i : A \times T_i \to \mathbb{R}$, where $A = A_1 \times A_2$. Given $t_i \in T_i$ and $a \in A$, bidder i's payoff is:

$$u_i(a; t_i) = \begin{cases} t_i - a_j & \text{if } a_i \neq B \text{ and } a_i > a_j, \\ t_i - B & \text{if } a_i = B \text{ and } a_i > a_j, \\ \frac{1}{2}(t_i - a_i) & \text{if } a_i = a_j, \\ 0 & \text{if } a_i < a_j. \end{cases}$$

If no one bids the buy price B, then the auction is the same as an ordinary second-price, sealed-bid auction. That is, the highest bidder obtains the item and pays the second highest bid (i.e., the other bidder's bid). If only one bidder bids the buy price B, then he obtains the item and must pay it to the seller. If both two bidders bid the same amount (It might be the buy price B.), we adopt the tie-breaking rule that a winner is determined with equal probability.

Bidder i's strategy is $\sigma_i : T_i \to \Delta(A_i)$, where $\Delta(A_i)$ is the set of probability distributions over A_i.[6] A solution concept is Bayesian Nash equilibrium: the strategy profile $\sigma = (\sigma_1(\cdot), \sigma_2(\cdot))$ is a Bayesian Nash equilibrium if for all $i \in N$, all $t_i \in T_i$, and all $a'_i \in A_i$,

$$E[u_i(a; t_i)|\sigma_i(\cdot), \sigma_j(\cdot), \rho(\cdot)] \geq E[u_i(a'_i, a_j; t_i)|\sigma_j(\cdot), \rho(\cdot)],$$

where $\sigma_j(\cdot)$ is the other bidder's strategy and $\rho(\cdot)$ is the probability distribution over T_j.

3 Results

We examine the effects of buy prices on seller's equilibrium revenues. To understand the influences, we contrast the case of the auctions without a buy price B and the case of the auctions with a buy price B. Specifically, we focus on the maximum (or supremum) and the minimum (or infimum) of equilibrium revenues. We analyze all equilibria.

3.1 The auctions without a buy price B

We consider the auctions without a buy price B. When we analyze all equilibria, an equilibrium revenue is not unique. Then, we pay much attention to the maximum (or supremum) and the minimum (or infimum) of equilibrium revenues.

First, we derive the minimum (or infimum) of equilibrium revenues. Indeed, the minimum of equilibrium revenues is zero. For example, consider the following strategy profile $\underline{\sigma} = (\underline{\sigma}_1(\cdot), \underline{\sigma}_2(\cdot))$:

$$\underline{\sigma}_1(t_1) = 0 \text{ for all } t_1 \text{ and } \underline{\sigma}_2(t_2) = v^H \text{ for all } t_2.$$

[5] We do not consider the choice of a buy price by the seller.

[6] When $\sigma_i(\cdot)$ is a pure strategy, we often regard the range of $\sigma_i(\cdot)$ as A_i.

The strategy profile $\underline{\sigma}$ is an equilibrium and the equilibrium revenue is zero.

Next, we derive the maximum (or supremum) of equilibrium revenues. Generally, to be an equilibrium, one of the bidders' v^L-types must submit a bid below v^L. It is because if both bidders' v^L-types submitted bids strictly above v^L with positive probability, one of the bidders could increase his payoff by undercutting his bid. Here, without loss of generality, let bidder 1 be the bidder whose v^L-type bids below v^L. Thus, the seller's revenues are at most v^L when bidder 1 is v^L-type.

For the same reason as the case in which both bidders are v^L-type, one of the bidders' v^H-types must submit a bid below v^H. Here let bidder 2 be the bidder whose v^H-type bids below v^H. In addition, bidder 2's v^L-type can bid strictly above v^L. Then, the seller's revenues are at most v^H when bidder 1 is v^H-type.[7]

From the above arguments, the maximum of equilibrium revenues is $pv^L + (1-p)v^H$. For example, consider the following strategy profile $\overline{\sigma} = (\overline{\sigma}_1(\cdot), \overline{\sigma}_2(\cdot))$:

$$\overline{\sigma}_1(t_1) = \begin{cases} b_{1H} \ (> v^H) & \text{if } t_1 = v^H, \\ t_1 & \text{if } t_1 = v^L \end{cases} \quad \text{and } \overline{\sigma}_2(t_2) = v^H \text{ for all } t_2.$$

The strategy profile $\overline{\sigma}$ is an equilibrium and the equilibrium revenue is $pv^L + (1-p)v^H$. The seller's expected revenue at the weakly dominant strategy equilibrium, where each type of bidders submits own valuation, is $p(2-p)v^L + (1-p)^2 v^H$. This equilibrium revenue is less than $pv^L + (1-p)v^H$.

Recall the strategy profile $\underline{\sigma}$. Replacing bidder 1's bids with $b_1 \in (0, v^L]$, the modified strategy profile is also an equilibrium and then the equilibrium revenue is $b_1 \in (0, v^L]$. Similarly, recall the strategy profile $\overline{\sigma}$. Replacing bidder 2's bids with $b_2 \in (v^L, v^H)$, the modified strategy profile is also an equilibrium and then the equilibrium revenue is $b_2 \in (v^L, pv^L + (1-p)v^H)$.

To summarize, we have the following proposition.

Proposition 1. *Consider the auctions without a buy price B. Then, the seller obtains an equilibrium revenue in* $[0, pv^L + (1-p)v^H]$.

3.2 The auctions with a buy price B

Now, we consider the auctions with a buy price $B \in (v^L, v^H]$. The auctions are roughly divided into two classes. One is that the seller sets a buy price $B \in (v^L, B^*]$. The other is that the seller sets a buy price $B \in (B^*, v^H]$.

Here we provide an explanation for the buy price B^*. When we consider the auctions, it is natural to focus on the symmetric strategy profile where bidder's v^H-type bids a buy price B and bidder's v^L-type bids own valuation. It is because a bidder whose type is greater than, or equal to the buy price B actually bids it and because a bidder whose type is less than the buy price B takes a weakly dominant action. This strategy profile is an equilibrium if and only if

$$p(v^H - B) + \frac{1-p}{2}(v^H - B) \geq p(v^H - v^L) \tag{1}$$

holds. The buy price B^* is the highest one that satisfies (1).[8][9]

We consider each class and then analyze all equilibria. As well as the case of the auctions without a buy price B, we pay much attention to the maximum (or supremum) and the minimum (or infimum) of equilibrium revenues.

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[7] If we let bidder 1 be the bidder whose v^H-type bids below v^H, we can only derive a supremum of equilibrium revenues. It is because bidder 2's v^L-type does not bid above v^H in this case.

[8] That is, (1) holds with equality. See Budish and Takeyama (2001) for details.

[9] We can always find a buy price $B \in (v^L, v^H]$ such that (1) holds.

3.2.1 The auctions with a buy price $B \in (v^L, B^*]$

We consider the case in which the seller sets a buy price $B \in (v^L, B^*]$. Fix a buy price B. At an equilibrium, in the same as the case of the auctions without a buy price B, both bidders' v^L-types do not bid strictly above v^L with positive probability. Without loss of generality, let bidder 1 be the bidder whose v^L-type bids below v^L.

If both bidders' v^H-types bid the amount except the buy price B with positive probability, one of the bidders could increase his payoff by bidding the amount that is slightly above the other bidder's bid. Therefore, only one of the bidders' v^H-types can randomize his action between bidding the buy price B and bidding the amount except the buy price B. Here we assume that the following condition holds:[10]

$$p(v^H - v^L) + \frac{1-p}{2}(v^H - v^L) \leq p(v^H - 0). \tag{2}$$

Then, there always exists $0 < b \leq v^L$ such that

$$p(v^H - B) + \frac{1-p}{2}(v^H - B) = p(v^H - b) \tag{3}$$

holds. Note that (3) holds for $b = v^L$ when $B = B^*$.

First, we derive the minimum (or infimum) of equilibrium revenues. At an equilibrium, bidder 2's v^H-type bids the amount except the buy price B with positive probability.[11] If each type of bidder 2 submits the amount except the buy price B and wins, then he pays the other bidder's bid (i.e., the bid of bidder 1's v^L-type). Here let bidder 1's v^L-type bid b with probability 1.[12]

Suppose that bidder 2's v^H-type bids $b_{2H} \neq B$ with probability β, and that bidder 2's v^L-type bids $b_{2L} \in [v^L, B)$. Since bidder 1's v^H-type bids the buy price B at an equilibrium,

$$\begin{aligned} &\beta\{p(v^H - B) + (1-p)(v^H - B)\} + (1-\beta)\left\{p(v^H - B) + \frac{(1-p)}{2}(v^H - B)\right\} \\ &\geq \beta\{p(v^H - b_{2L}) + (1-p)(v^H - b_{2H})\} + (1-\beta)p(v^H - b_{2L}) \end{aligned} \tag{4}$$

must hold. The LHS of (4) is the expected payoff that bidder 1's v^H-type obtains by bidding the buy price B. The RHS of (4) is the maximum of expected payoffs that bidder 1's v^H-type obtains by bidding the amount except the buy price B. Calculating (4), we have

$$\beta \leq \frac{\frac{(1+p)}{2}(v^H - B) - p(v^H - b_{2L})}{(1-p)(v^H - b_{2H}) - \frac{(1-p)}{2}(v^H - B)}, \tag{5}$$

which is less than 1.[13]

From the above arguments, the equilibrium revenue is given by

$$R = p^2 b + p(1-p)\big\{\beta b + (2-\beta)B\big\} + (1-p)^2 B. \tag{6}$$

By (5), we can derive the supremum of β, which is 1.[14] Then, the infimum of equilibrium revenues is given by

$$\underline{R} = pb + (1-p)B. \tag{7}$$

By (3), b is monotone increasing with respect to the buy price B. Then, (7) is monotone increasing with respect to the buy price B.

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[10] In the Appendix B, we consider the case in which (2) does not hold. Even in this case, we still obtain the main result.

[11] Indeed, it suffices to consider the case in which bidder 1's v^H-type bids the buy price B with probability 1. Our argument below includes the case in which bidder 1's v^H-type randomizes his action.

[12] One might think that to lower an equilibrium revenue, bidder 1's v^L-type randomizes his action such that (3) holds. We argue this issue in the Appendix C.

[13] In the Appendix A, we prove that β is less than 1.

[14] We evaluate (5) at $b_{2L} \approx B$ and $b_{2H} \approx B$.

Proposition 2. *The infimum of equilibrium revenues in the auctions with a buy price B is monotone increasing with respect to the buy price B.*

Next, we derive the maximum (or supremum) of equilibrium revenues. In the same as the case of auctions without a buy price B, the seller's revenue is at most v^L when both bidders are v^L-type. To maximize an equilibrium revenue, each bidder's v^H-type must submit a buy price B with probability 1. Therefore, the maximum of equilibrium revenues is given by

$$\overline{R} = p^2 v^L + (1-p^2)B. \tag{8}$$

Recall the derivation of (7). Since $\beta \in [0,1)$, the equilibrium revenue is in $(pb + (1-p)B, p^2 b + (1-p^2)B]$. Next, recall the derivation of (8). We have considered the symmetric strategy equilibrium where bidder's v^H-type bids the buy price B and bidder's v^L-type bids his own valuation. Here we consider the strategy profile where both bidders' v^H-types bid the buy price B, one of the bidders' v^L-types bid his own valuation, but one of the bidders' v^L-types submits the bid in $[b, v^L)$. Without loss of generality, let bidder 1 be the bidder whose v^L-type submits the bid in $[b, v^L)$. Then, bidder 2's v^L-type takes a weakly dominant action. The incentive constraint of bidder 1's v^H-type does not change and thus holds. By (3), the incentive constraint of bidder 2's v^H-type also holds. Finally, bidder 1's v^L-type does not have a profitable deviation. Therefore, such a strategy profile is also an equilibrium and then the equilibrium revenue is in $[p^2 b + (1-p^2)B, p^2 v^L + (1-p^2)B)$.

To summarize, we have the following proposition.

Proposition 3. *Consider the auctions with a buy price $B \in (v^L, B^*]$. Then, the seller obtains an equilibrium revenue in $(pb + (1-p)B, p^2 v^L + (1-p^2)B]$.*

3.2.2 The auctions with a buy price $B \in (B^*, v^H]$

We consider the case in which the seller sets a buy price $B \in (B^*, v^H]$. Fix a buy price B. Then, there exists $b > v^L$ such that (3) holds. Since both bidders' v^L-types do not bid strictly above v^L with positive probability at an equilibrium, without loss of generality, let bidder 1 be the bidder whose v^L-type can bid above v^L.

First, we consider the strategy profiles where both bidders' v^H-types bid the buy price B with probability 1. In this case, the incentive constraint of bidder 1's v^H-type does not hold because bidder 2's v^L-type must bid below v^L at an equilibrium.

Next, we consider the strategy profiles where one of the bidders' v^H-types randomizes his action between bidding the buy price B and bidding the amount except the buy price B. In this case, from a similar argument to that of Subsubsection 3.2.1, bidder 2's v^H-type can randomize his action. To guarantee the existence of such an equilibrium,

$$\begin{aligned}&\beta\{p(v^H - B) + (1-p)(v^H - B)\} + (1-\beta)\left\{p(v^H - B) + \frac{(1-p)}{2}(v^H - B)\right\}\\ &\quad > \beta\{p(v^H - v^L) + (1-p)(v^H - B)\} + (1-\beta)p(v^H - v^L)\end{aligned} \tag{9}$$

must hold because bidder 2's v^H-type bids the amount except the buy price B with positive probability and because bidder 2's v^L-type bids at most v^L at an equilibrium. However, (9) clearly does not hold.

We have already mentioned that such a strategy profile where each bidder's v^H-type bids the amount except the buy price B with positive probability is not an equilibrium. Therefore, we have the following proposition.

Proposition 4. *Consider the auctions with a buy price $B \in (B^*, v^H]$. Then, there is no equilibrium.*

By Proposition 3 and 4, we have the main result.

Theorem 1. *A seller can always obtain a positive equilibrium revenue by introducing a buy price B.*

4 Conclusion

We have investigated how the introduction of buy prices affects the seller's equilibrium revenues. We have shown that an equilibrium revenue is always positive in the auctions with a buy price, while it can be zero in the auctions without a buy price. This might be a reason why a lot of sellers actually introduce a buy price in Internet auctions.

Appendix A

β is less than 1.

We prove by contradiction that β is less than 1. Suppose that

$$\frac{\frac{(1+p)}{2}(v^H - B) - p(v^H - b_{2L})}{(1-p)(v^H - b_{2H}) - \frac{(1-p)}{2}(v^H - B)} > 1. \tag{10}$$

Calculating (10), we have

$$(v^H - B) > p(v^H - b_{2L}) + (1-p)(v^H - b_{2H}),$$

which does not hold. This contradicts the assumption.

Appendix B

The case: (2) does not hold.

We consider the case in which (2) does not hold. There does not exist $0 \leq b$ such that (3) holds under the buy price B which is less than or equal to a certain threshold. Therefore, in these cases, no bidder's v^H-type can randomize his action at an equilibrium. Then, the minimum of equilibrium revenues is given by $(1-p^2)B$. The minimum of equilibrium revenues is also increasing with respect to the buy price B.

Now we consider the least buy price B under which there exists $b = 0$ such that (3) holds. In this case, the infimum of equilibrium revenues is $(1-p)B$, which is less than $(1-p^2)B$. Therefore, the minimum (or infimum) of equilibrium revenues jumps down at a certain buy price B. In other words, we cannot obtain Proposition 2. However, we still obtain the main result.

Appendix C

The case: bidder 1's v^L-type randomizes his action.

We consider the case in which bidder 1's v^L-type randomizes his action. Suppose that bidder 1's v^L-type bids $0 \leq \underline{b}_{1L} < b$ (resp. $b < \overline{b}_{1L}$) with probability α (resp. $1-\alpha$). Since (3) must hold, α is determined as follows:

$$b = \alpha \underline{b}_{1L} + (1-\alpha)\overline{b}_{1L}. \tag{11}$$

Calculating (11), we have

$$\alpha = \frac{\overline{b}_{1L} - b}{\overline{b}_{1L} - \underline{b}_{1L}}.$$

Indeed, it is classified into two cases.

Case 1: $\bar{b}_{1L} \leq v^L$.
In this case, the equilibrium revenue is given by

$$R = p^2\{\alpha\underline{b}_{1L} + (1-\alpha)\bar{b}_{1L}\} + p(1-p)\big[\beta\{\alpha\underline{b}_{1L} + (1-\alpha)\bar{b}_{1L}\} + (2-\beta)B\big] \\ + (1-p)^2 B.$$

Since $b = \alpha\underline{b}_{1L} + (1-\alpha)\bar{b}_{1L}$ for all pairs $(\underline{b}_{1L}, \bar{b}_{1L})$, it suffices to consider (7).

Case 2: $v^L < \bar{b}_{1L}$.
When bidder 2's v^L-type bids v^L, bidder 1's v^L-type wins the auction with probability $1-\alpha$ and pays v^L to the seller. Therefore, the equilibrium revenue is given by

$$R = p^2\{\alpha\underline{b}_{1L} + (1-\alpha)v^L\} + p(1-p)\big[\beta\{\alpha\underline{b}_{1L} + (1-\alpha)\bar{b}_{1L}\} + (2-\beta)B\big] \\ + (1-p)^2 B. \tag{12}$$

Given β. Then, plugging $\alpha = (\bar{b}_{1L} - b)/(\bar{b}_{1L} - \underline{b}_{1L})$ into (12) and partially differentiating with respect to $\underline{b}_{1L}$, we have

$$\frac{p^2(\bar{b}_{1L} - b)(\bar{b}_{1L} - v^L)}{(\bar{b}_{1L} - \underline{b}_{1L})^2} > 0.$$

Similarly, partially differentiating with respect to $\bar{b}_{1L}$, we have

$$-\frac{p^2(b - \underline{b}_{1L})(v^L - \underline{b}_{1L})}{(\bar{b}_{1L} - \underline{b}_{1L})^2} < 0.$$

Therefore, we obtain the infimum of (12) at $(\underline{b}_{1L}, \bar{b}_{1L}) = (0, B)$. Note that bidder 1's v^L-type actually cannot bid B. In this case,

$$\alpha^* = \frac{B - b}{B}. \tag{13}$$

Next, we derive the supremum of β. By (5),

$$\beta^* = \frac{(1+p)(v^H - B) - 2p(v^H - v^L)}{(1-p)(v^H - B)}. \tag{14}$$

Note that bidder 2's v^L-type must bit v^L at an equilibrium. Therefore, the infimum of equilibrium revenues is given by

$$R = p^2(1-\alpha^*)v^L + p(1-p)\{\beta^* b + (2-\beta^*)B\} + (1-p)^2 B. \tag{15}$$

Now, we examine when (7) is the infimum of equilibrium revenues in all cases. It suffices to consider when (15) > (7). By (3),

$$b = v^H - \frac{(1+p)}{2p}(v^H - B). \tag{16}$$

Thus, by (13), (14), and (16), we have

$$(15) - (7) = \frac{p(B - v^L)}{2B}\{(1-p)v^H + (1-3p)B\}.$$

Therefore, to go through with the argument in the text, we need to check additionally whether the following condition holds:

$$\begin{array}{cl} (1-p)v^H + (1-3p)v^L \geq 0 & \text{if } p < \frac{1}{3}, \\ \text{no condition} & \text{if } p = \frac{1}{3}, \\ (1-p)v^H + (1-3p)B^* \geq 0 & \text{if } p > \frac{1}{3}. \end{array}$$

Note that even if the condition does not hold, by (15), we still obtain the main result.

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