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The Evolution of Fairness under an Assortative Matching Rule in the Ultimatum Game

by

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Abstract

This paper studies how a matching rule affects the evolution of fairness in an ultimatum mini game. Gale et al. [1995] show that only selfish behaviour survives in the deterministic replicator dynamics under the random matching rule. In contrast, this paper shows that, under an assortative matching rule, the fair behaviour may survive at an asymptotically stable state.

1 Introduction

Why would people behave in a fair manner, sacrificing their own monetary payoffs. In the ultimatum game, selfish individuals propose to exploit almost the total surplus, and accept these unfair offers as responders. Contrary to this standard game theoretical prediction, many experimental data show that people tend to divide the total surplus equally (e.g. Güth et al. [1982] and Binmore et al. [2002]). This paper studies this paradox in the framework of the evolutionary game theory, focusing on a matching rule.

Gale et al. [1995] consider the ultimatum mini game given in Figure 1. In the game, agent 1 proposes either a high offer (H) or a low offer (L). If she adopts strategy H, it is assumed that agent 2 (responder) always accepts it. If she adopts strategy L, the responder decides to accept (Y) or reject (N) it. Let x_1 and x_2 be proportions of individuals adopting actions L and Y, respectively, in the population of proposers and responders, respectively.

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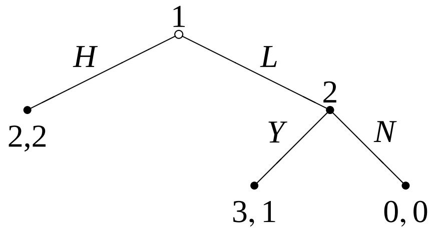

Figure 1: The ultimatum mini game.

The standard replicator dynamics is described as

$$\begin{aligned} \dot{x}_1 &= g_1(x) = x_1(f_L - \phi_1) \\ \dot{x}_2 &= g_2(x) = x_2(f_Y - \phi_2), \end{aligned}$$

where f_k is the fitness of individuals adopting action k ($k = L, Y$), and ϕ_i is the average fitness of population $i = 1, 2$. In the ultimatum mini game given in Figure 1, it holds that

$$\begin{aligned} g_1(x) &= x_1(1 - x_1)(3x_2 - 2) \\ g_2(x) &= x_2(1 - x_2)x_1. \end{aligned}$$

Gale et al. [1995] show that it is a unique asymptotically stable state of the deterministic replicator dynamics that all proposers will make the selfish offer (L) and all responders will accept it (Y). That is, only the unfair behaviour survives in the replicator dynamics under the random matching rule. Thus, the paradox still holds.

To represent an evolutionary drift, Gale et al. [1995] and Binmore and Samuelson [1999] introduce the following perturbed selection dynamics,

$$\dot{x} = g(x) + h(x).$$

If a drift function h is strictly decreasing in a difference between the largest and the smallest expected payoffs, there exists an asymptotically stable state which is an imperfect Nash equilibrium leading to the fair allocation. The asymptotically stable state, however, critically depends on the form of h (Binmore and Samuelson [1999]). For example, if h is not sensitive to payoffs, then only the subgame perfect equilibrium is asymptotically stable.

Unlike these previous approaches, this paper studies how a matching rule affects the evolution of fairness. Especially, we consider an assortative matching rule which is introduced by Becker [1973, 1974]. The assortative matching rule is a matching rule under which similar types of individuals are paired more often than under the random matching rule.

Under the assortative matching rule, an interaction rate between individuals depends on their own actions in contrast to the random matching rule. This property leads to the replicator dynamics with non-linear fitness functions (Taylor and Nowak [2006]). In symmetric 2×2 strategic form games, Taylor and Nowak [2006] introduce a generalized matching rule with non-uniform interaction rates. They show that the non-uniform interaction rates generate interior equilibria even if one strategy dominates another. Bergstrom [2003] introduces another kind of the assortative matching rule in the prisoners' dilemma game. Taylor and Nowak [2006] and Bergstrom [2003] show that cooperation survives under the assortative matching rule in the prisoners' dilemma game.

The main result of this paper is that, under an assortative matching rule, there exist asymptotically stable states at which the fair allocation may prevail. Especially, if the matching rule is completely assortative, there exist only two asymptotically stable states, the fully fair equilibrium and selfish equilibrium. The results provide an evolutionary support for the fair allocation which has been observed by many experiments in the ultimatum game.

The paper is organized as follows. Section 2 defines an assortative matching rule and the selection dynamics of the ultimatum mini game. Section 3 presents the main results. Section 4 gives an example of an assortative matching rule. Section 5 discusses the results.

2 The Model

There are two populations, proposers (population 1) and responders (population 2). The sizes of the two populations are equal. In each period, an agent in one population matches with an agent in the other population according to a predetermined matching rule. Each pair of agents plays the ultimatum mini game given in Figure 1.

We call a proposer selfish if she adopts strategy L, and fair if she adopts H. Similarly, we call a responder selfish if he adopts strategy Y, and fair if he adopts N. Let x_1 denote the proportion of selfish proposers in population 1, and x_2 denote the proportion of selfish responders in population 2, respectively. A state of the system is represented by a pair $x = (x_1, x_2)$. For a technical reason, we assume that the range of x_i is $[\epsilon, 1 - \epsilon]$ for all $i = 1, 2$ where ϵ is a sufficiently small positive number [1].

[1] If this assumption does not hold, the matching rule $(p_i, q_i)_{i=1,2}$ are not Lipschitz continuous (see Definition 1). Section 3 and 4 discuss the set of asymptotically stable states as ϵ goes to zero.

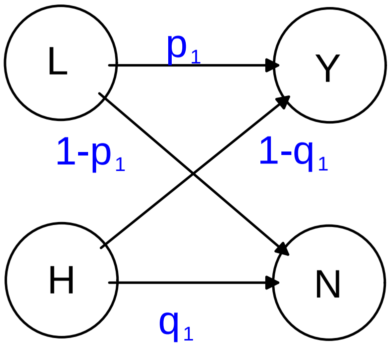

Figure 2: The matching probability in population 1: $p_1 = Pr(L \text{ meets } Y)$, $q_1 = Pr(H \text{ meets } N)$.

Now, we define a matching rule as a pair $(p_i(x), q_i(x))_{i=1,2}$ of two functions on $[\epsilon, 1-\epsilon]\times[\epsilon, 1-\epsilon]$, where $p_i(x)$ is the probability that a selfish agent in population i meets a selfish agent in population j and $q_i(x)$ is the probability that a fair agent in population i meets a fair agent in population j at state x (Figure 2).

Definition 1. $(p_i(x), p_j(x))_{i=1,2}$ is a matching rule if for all $i = 1, 2$, $p_i(x)$ and $q_i(x)$ are Lipschitz continuous on $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$ [2], and satisfies

$$\lim_{x_j\to 0} p_i(x) = 0, \qquad \lim_{x_j\to 1} p_i(x) = 1, \qquad \lim_{x_i\to 1} p_i(x) = x_j,$$
$$\lim_{x_j\to 1} q_i(x) = 0, \qquad \lim_{x_j\to 0} q_i(x) = 1, \qquad \lim_{x_i\to 0} q_i(x) = 1 - x_j,$$

and for all $x \in [\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$,

$$x_i p_i = x_j p_j, \tag{1}$$
$$(1 - x_i)q_i = (1 - x_j)q_j, \tag{2}$$
$$(1 - p_i)x_i = (1 - q_j)(1 - x_j). \tag{3}$$

Equations (1), (2), and (3) are parity equations which imply that probability functions p_i and q_i are consistent as a matching rule. All players can be paired as long as these equations are satisfied. Note that if one of p_1, p_2, q_1, or q_2 is determined, then the other valuables is automatically determined by using these conditions. The random matching rule assumes that $p_i(x) = q_i(x) = x_j$ for all i and j $(i \neq j)$.

Under an assortative matching rule, similar types of agents are matched more often than under the random matching rule.

[2] The function p is Lipschitz continuous if for any $x, y \in [\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$, there exists a constant k such that $\mid p(y) - p(x) \mid < k \mid y - x \mid$.

Definition 2. A matching rule $(p_i, q_i)_{i=1,2}$ is *assortative*, if for all $i = 1, 2$, p_i and q_i satisfy the following conditions:

(i) $p_i(x)$ is monotonically non-increasing in x_i and monotonically non-decreasing in x_j.
$q_i(x)$ is monotonically non-increasing in x_j and monotonically non-decreasing in x_i.

(ii) $p_i(x) > x_j$ and $q_i(x) > 1 - x_j$.

(iii) $\lim_{x_i \to 0} p_i(x) > x_j$ and $\lim_{x_i \to 1} q_i(x) > 1 - x_j$.

An assortative matching rule has two characteristics. First, by (i) and (ii), the probability p_i has a strictly higher value than the one under the random matching rule and it increases as the frequency of their same type opponents increases. Second, the increase of the selfish type frequency x_i causes the decrease of the probability p_i. Let $\mathcal{AM}$ be the set of assortative matching rules.

Next, we define the replicator dynamics, where the state space is $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$.

Definition 3. The replicator dynamics with a matching rule $(p_i, q_i)_{i=1,2}$ on $[\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$ is defined by

$$\dot{x}_1 = (x_1 - \epsilon)(f_L - \phi_1) \tag{4}$$
$$\dot{x}_2 = (x_2 - \epsilon)(f_Y - \phi_2), \tag{5}$$

where ϕ_i is the average payoff of population i such that

$$\phi_1 = (x_1 + \epsilon) f_L + (1 - x_1 - \epsilon) f_H$$
$$\phi_2 = (x_2 + \epsilon) f_Y + (1 - x_2 - \epsilon) f_N,$$

and f_j $(j = L, H, Y, N)$ is the expected payoff for strategy j; $f_L = 3p_1(x)$, $f_H = 2$, $f_Y = p_2(x) + 2(1 - p_2(x))$, and $f_N = 2q_2(x)$.

Thus, the system is described as

$$\dot{x}_1 = g_1(x) = (x_1 - \epsilon)(1 - \epsilon - x_1)\overline{\phi}_1(x) \tag{6}$$
$$\dot{x}_2 = g_2(x) = (x_2 - \epsilon)(1 - \epsilon - x_2)\overline{\phi}_2(x), \tag{7}$$

where

$$\overline{\phi}_1(x) = 3p_1(x) - 2$$
$$\overline{\phi}_2(x) = 2 - p_2(x) - 2q_2(x).$$

Although this dynamics is not the standard replicator dynamics, it has the same properties, regularity and monotonicity (Binmore and Samuelson [1999]). In this selection dynamics, the growth rate is continuous on state space $[\epsilon, 1-\epsilon]\times[\epsilon, 1-\epsilon]$ (regularity) and that a growth rate of a relatively low-payoff action is smaller than that of a relatively high-payoff action (monotonicity).

Finally, we define some standard concepts of dynamic stability (e.g. Vega-Redondo [2003]).

Definition 4. (1) A state $x = (x_1, x_2)$ is a *rest point* of (4) and (5) if and only if $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

(2) A state x^* is an *asymptotically stable point* of (4) and (5) if and only if the following two conditions hold:

(i) (Liapnov stability) Given any neighborhood U_1 of x^*, there exists some neighborhood U_2 of x^*, such that for any path $x = x(t)$, $x(0) \in U_2$ implies $x(t) \in U_1$ for all $t > 0$.

(ii) There exists some neighborhood V of x^* such that for any path $x = x(t)$, $x(0) \in V$ implies $\lim_{t\to\infty} x(t) = x^*$.

3 Results

Proposition 1. *Let $\mathcal{R}(a)$ be the set of all rest points of system* (6) *and* (7) *under an assortative matching rule $a = (p_i, q_i)_{i=1,2}$. As ϵ goes to 0, $\mathcal{R} = \bigcup_{a\in\mathcal{AM}} \mathcal{R}(a)$ converges to*

$$\{(1, 0), (1, 1), (x_1', x_2')\} \cup \{(0, c)|c \in [0, 1]\},$$

where (x_1', x_2') satisfies $\bar{\phi}_1(x_1', x_2') = 0$ and $\bar{\phi}_2(x_1', x_2') = 0$.

Proof. Obviously, $(\epsilon, \epsilon), (\epsilon, 1-\epsilon), (1-\epsilon, \epsilon), (1-\epsilon, 1-\epsilon)$ are rest points of (6) and (7). Next, we show that there exists no $a \in \mathcal{AM}$ under which there exists some x_1 satisfying $\bar{\phi}_1(x_1, 1-\epsilon) = 0$. Suppose that $\bar{\phi}_1(x_1, 1-\epsilon) = 0$. Then, $p_1(x_1, 1-\epsilon) = \frac{2}{3}$. This implies $p_2(x_1, 1-\epsilon) = \frac{2}{3(1-\epsilon)} x_1$ by parity equations (1) and (3). However, this contradicts condition (ii) in Definition 2 ($p_2 > x_1$) when ϵ is sufficiently small.

We can construct an assortative matching rule under which there exist x_1', x_2', x_1'', x_2'', and x_2''' satisfying $\bar{\phi}_1(x_1', x_2') = 0$, $\bar{\phi}_2(x_1', x_2') = 0$, $\bar{\phi}_1(x_1'', \epsilon) = 0$, $\bar{\phi}_2(\epsilon, x_2'') = 0$, and $\bar{\phi}_2(1-\epsilon, x_2''') = 0$. Define an assortative matching rule $a' = (p_i, q_i)_{i=1,2}$ as

$$p_1(x_1, x_2) = (1-\alpha)x_2 + \alpha(\min[\frac{x_2}{x_1}, 1]), \tag{8}$$

where $\alpha \in (0, 1]$. Under this a', when $\alpha = 1$,[3] (x_1', x_2'), (x_1'', ϵ), $(1-\epsilon, x_2''')$, (x_1', x_2') exist, and when $\alpha = 1/2$, (ϵ, x_2'') exists.

By (6) and (7), there exist no other rest points under any assortative matching rule. Thus, for sufficiently small ϵ,

$$\begin{aligned}\mathcal{R} = &\{(\epsilon, \epsilon), (\epsilon, 1-\epsilon), (1-\epsilon, \epsilon), (1-\epsilon, 1-\epsilon)\}\\ &\cup\{(x_1', x_2'), (x_1'', \epsilon), (\epsilon, x_2''), (1-\epsilon, x_2''')\}.\end{aligned}$$

Since (1), $\overline{\phi}_1(x_1'', \epsilon) = 0$, and $p_2 \le 1$, we obtain

$$x_1'' = \frac{\epsilon}{p_1(x_1'', \epsilon)} p_2(x_1'', \epsilon) \le \frac{3}{2}\epsilon.$$

Then, x_1'' converges to 0 as ϵ goes to 0. Since $\overline{\phi}_2(1-\epsilon, x_2''') = 0$ and $p_2, q_1 \le 1$, we obtain

$$\begin{aligned} 2 &= p_2(1-\epsilon, x_2''') + 2q_2(1-\epsilon, x_2''') \\ &= p_2(1-\epsilon, x_2''') + 2q_1(1-\epsilon, x_2''')\frac{\epsilon}{1-x_2'''} \\ &\le 1 + 2\frac{\epsilon}{1-x_2'''}. \end{aligned}$$

Then, $x_2''' \ge 1-2\epsilon$. Hence, x_2''' converges to 1 as ϵ goes to 0. Since $\lim_{\epsilon\to 0} p_2(\epsilon, x_2'') = 0$ and $\lim_{\epsilon\to 0} q_2(\epsilon, x_2'') = 1$, any $x_2'' \in [\epsilon, 1-\epsilon]$ satisfies $\lim_{\epsilon\to 0} \overline{\phi}_2(\epsilon, x_2'') = 0$.

Thus, as ϵ goes to 0, $\mathcal{R}$ converges to

$$\{(1, 0), (1, 1), (x_1', x_2')\} \cup \{(0, x_2) \mid x_2 \in [0, 1]\}.$$

□

Since any x with $x_1 = 0$ is in $\mathcal{R}$, the equal allocation is supported as a rest point of the replicator dynamics (6)-(7). The following proposition shows that the equal allocation is asymptotically stable.

Proposition 2. *Let $\mathcal{A}(a)$ be the set of asymptotically stable points under an assortative matching rule $a = (p_i, q_i)_{i=1,2}$. As ϵ goes to 0, $\mathcal{A} = \bigcup_{a\in\mathcal{AM}} \mathcal{A}(a)$ converges to*

$$\{(1, 1)\} \cup \{(0, x_2) | x_2 \in [0, 1/2)\}.$$

Proof. We first prove the two claims.

[3] This rule is equal to a rule in section 4.

Claim 1. *There exists no assortative matching rule under which either* $(\epsilon, 1-\epsilon)$ *or* $(1-\epsilon, \epsilon)$ *is asymptotically stable.*

We will show only $x = (\epsilon, 1-\epsilon) \notin \mathcal{A}$. The Jacobian matrix of g at $(\epsilon, 1-\epsilon)$ is

$$\begin{aligned}
&\frac{\partial g}{\partial x}(\epsilon, 1-\epsilon) \\
&= \begin{pmatrix} (1-2\epsilon)(3p_1(\epsilon, 1-\epsilon)-2) & 0 \\ 0 & (2\epsilon-1)(2-p_2(\epsilon, 1-\epsilon)-2q_2(\epsilon, 1-\epsilon)) \end{pmatrix} \\
&= \begin{pmatrix} (1-2\epsilon)(3p_1(\epsilon, 1-\epsilon)-2) & 0 \\ 0 & (2\epsilon-1)(2-\frac{2-\epsilon}{1-\epsilon}p_1(\epsilon, 1-\epsilon)) \end{pmatrix}.
\end{aligned}$$

It is well-known that a rest point of the system is asymptotically stable if and only if the real parts of both eigenvalues of the Jacobian are negative (e.g. Arnold [2006]). Thus, $(\epsilon, 1-\epsilon)$ is asymptotically stable if and only if $\frac{2(1-\epsilon)}{2-\epsilon} < p_1(\epsilon, 1-\epsilon) < \frac{2}{3}$. Since $p_1(\epsilon, 1-\epsilon) > 1-\epsilon$ by (ii) in Definition 2, $p_1(\epsilon, 1-\epsilon) < \frac{2}{3}$ is impossible for sufficiently small ϵ. We can show that $(1-\epsilon, \epsilon)$ is not asymptotically stable by the same procedure.

Claim 2. *If there exists* $x' = (x_1', x_2')$ *which satisfies* $\bar{\phi}_1(x') = 0$ *and* $\bar{\phi}_2(x') = 0$, *then* x' *is a saddle point under any assortative matching rule.*

Since $\bar{\phi}_1(x') = 0$, $p_1(x') = \frac{2}{3}$. Then, $p_2(x') = \frac{2x_1'}{3x_2'}$ and $q_2(x') = 1 - \frac{x_1'}{3(1-x_2')}$ by the (1)-(3). Thus, we obtain $x_2' = \frac{1}{2}$ by $\bar{\phi}_2(x') = 0$. The Jacobian matrix of g at (x_1', x_2') is

$$\begin{aligned}
\frac{\partial g}{\partial x}(x_1', x_2') &= \begin{pmatrix} (x_1'(1-x_1') - \epsilon(1-\epsilon))(\frac{\partial \bar{\phi}_1(x')}{\partial x_1}) & (x_1'-\epsilon)(1-\epsilon-x_1')(\frac{\partial \bar{\phi}_1(x')}{\partial x_2}) \\ (x_2'-\epsilon)(1-\epsilon-x_2')(\frac{\partial \bar{\phi}_2(x')}{\partial x_1}) & (x_2'(1-x_2') - \epsilon(1-\epsilon))(\frac{\partial \bar{\phi}_2(x')}{\partial x_2}) \end{pmatrix} \\
&= \begin{pmatrix} (x_1'-\epsilon)(1-\epsilon-x_1')(3\frac{\partial p_1}{\partial x_1}(x')) & (x_1'-\epsilon)(1-\epsilon-x_1')(3\frac{\partial p_1}{\partial x_2}(x')) \\ (\frac{1}{2}-\epsilon)^2(-6x_1'\frac{\partial p_1}{\partial x_1}(x')) & (\frac{1}{2}-\epsilon)^2(x_1'(\frac{16}{3} - 6\frac{\partial p_1}{\partial x_2}(x'))) \end{pmatrix}.
\end{aligned}$$

In this matrix, one eigenvalue is negative and the other is positive. Therefore, (x_1', x_2') is a saddle point.

By claims 1 and 2, it suffices us to construct assortative matching rules under which $x = (x_1'', \epsilon)$, (ϵ, x_2''), $(1-\epsilon, x_2''')$, (ϵ, ϵ), and $(1-\epsilon, 1-\epsilon)$ are asymptotically stable for sufficiently small ϵ. First, we consider $x = (\epsilon, \epsilon)$. The Jacobian matrix of g at (ϵ, ϵ) is

$$\begin{aligned}
\frac{\partial g}{\partial x}(\epsilon, \epsilon) &= \begin{pmatrix} (1-2\epsilon)(3p_1(\epsilon, \epsilon)-2) & 0 \\ 0 & (1-2\epsilon)(2-p_2(\epsilon, \epsilon)-2q_2(\epsilon, \epsilon)) \end{pmatrix} \\
&= \begin{pmatrix} (1-2\epsilon)(3p_1(\epsilon, \epsilon)-2) & 0 \\ 0 & (1-2\epsilon)(\frac{2\epsilon}{1-\epsilon} - p_1(\epsilon, \epsilon)\frac{1+\epsilon}{1-\epsilon}) \end{pmatrix}
\end{aligned}$$

by (1)-(3). If an assortative matching rule satisfies $\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon, \epsilon) < \frac{2}{3}$, then both eigenvalues are negative. If we consider the rule (8) with $\alpha = 1/10$, these conditions are satisfied. Hence, (ϵ, ϵ) is asymptotically stable if these conditions are satisfied. Assortative matching rules, under which other points, (x_1'', ϵ), $(1-\epsilon, 1-\epsilon)$, $(1-\epsilon, x_2''')$, and (ϵ, x_2'') are asymptotically stable for sufficiently small ϵ, are calculated by the same procedure. Therefore, we obtain

$$\mathcal{A} = \{(\epsilon, \epsilon), (1-\epsilon, 1-\epsilon), (x_1'', \epsilon), (1-\epsilon, x_2'''), (\epsilon, x_2'')\},$$

where x_1'', x_2'', and x_2''' satisfy $\bar{\phi}_1(x_1'', \epsilon) = 0$, $\bar{\phi}_2(\epsilon, x_2'') = 0$, and $\bar{\phi}_2(1-\epsilon, x_2''') = 0$.

For each $x \in \mathcal{A}$, the conditions that x is asymptotically stable for sufficiently small ϵ are given by the followings:

(i) (ϵ, ϵ) is asymptotically stable if an assortative matching rule satisfies

$$\frac{2\epsilon}{1+\epsilon} < p_1(\epsilon, \epsilon) < 2/3.$$

(ii) $(1-\epsilon, 1-\epsilon)$ is asymptotically stable if an assortative matching rule satisfies

$$\frac{2}{3} < p_1(1-\epsilon, 1-\epsilon) < 1 - \frac{\epsilon}{2-\epsilon}.$$

(iii) (x_1'', ϵ) is asymptotically stable if an assortative matching rule satisfies

$$\frac{\partial p_1}{\partial x_1}(x_1'', \epsilon) < 0$$
$$\frac{2\epsilon}{1+\epsilon} < p_1(x_1'', \epsilon) = \frac{2}{3}.$$

(iv) $(1-\epsilon, x_2''')$ is asymptotically stable if an assortative matching rule satisfies

$$\frac{2}{3} < p_1(1-\epsilon, x_2''') = 2\frac{x_2'''}{1+x_2'''}$$
$$\frac{\partial p_1}{\partial x_2}(1-\epsilon, x_2''') > 2\frac{1-x_2'''}{1+x_2'''}.$$

(v) (ϵ, x_2'') is asymptotically stable if an assortative matching rule satisfies

$$\frac{2}{3} > p_1(\epsilon, x_2'') = 2\frac{x_2''}{1+x_2''}$$
$$\frac{\partial p_1}{\partial x_2}(\epsilon, x_2'') > 2\frac{1-x_2''}{1+x_2''}.$$

By Proposition 1, $\lim_{\epsilon\to 0}(x_1'', \epsilon) = (0,0)$ and $\lim_{\epsilon\to 0}(1-\epsilon, x_2''') = (1,1)$. By parity equations (1)-(3), $\bar{\phi}_2(x) = 0$ implies

$$p_1(x_1, x_2) = 2\frac{x_2}{1+x_2}.$$

Since $p_1(\epsilon, x_2'') < \frac{2}{3}$, $x_2'' < 1/2$. Then, $\lim_{\epsilon\to 0}(\epsilon, x_2) \in \{(0,c)|c \in [0,1/2)\}$. Thus, as ϵ goes to 0, $\mathcal{A}$ converges to $\{(1,1)\} \cup \{(0,x_2)|x_2 \in [0,1/2)\}$. □

Proposition 2 shows that imperfect Nash equilibria in $(0,c)$, $c \in [0,1/2)$, are asymptotically stable under some assortative matching rules. We call each $(0,c)$ with $c \in (0,1/2)$ a *partially fair equilibrium* where all proposers are fair but there exist some selfish responders, $(0,0)$ the *fully fair equilibrium* where all agents choose fair actions, and $(1,1)$ the *selfish equilibrium* where all agents choose selfish actions.

Proposition 3. *The fully fair equilibrium and the selfish equilibrium are asymptotically stable under any assortative matching rule satisfying* $\frac{\partial p_1}{\partial x_1}(x_1'', \epsilon) < 0$ *where* $\bar{\phi}_2(\epsilon, x_2'') = 0$.

Proof. When $p_1(\epsilon,\epsilon) < 2/3$, (ϵ,ϵ) is asymptotically stable since $\frac{2\epsilon}{1+\epsilon} < \epsilon < p_1(\epsilon,\epsilon)$. When $p_1(\epsilon,\epsilon) \geq 2/3$, by $\frac{\partial p_1}{\partial x_1}(x_1'', \epsilon) < 0$, (x_1'', ϵ) is asymptotically stable. Thus, the fully fair equilibrium $(0,0)$ is asymptotically stable under any assortative matching rule satisfying $\frac{\partial p_1}{\partial x_1}(x_1'', \epsilon) < 0$, while a partially fair equilibrium is not always asymptotically stable[4].

When $1 - \frac{\epsilon}{2-\epsilon} > p_1(1-\epsilon, 1-\epsilon)$, $(1-\epsilon, 1-\epsilon)$ is asymptotically stable since $p_1(1-\epsilon, 1-\epsilon) > 1-\epsilon > 2/3$. When $1 - \frac{\epsilon}{2-\epsilon} \leq p_1(1-\epsilon, 1-\epsilon)$, since $q_2(x)$ is non-decreasing in x_2,

$$\frac{\partial p_1}{\partial x_2}(1-\epsilon, x_2''') \geq \frac{1 - p_1(1-\epsilon, x_2''')}{1 - x_2'''} = \frac{1}{1+x_2'''},$$

where $\bar{\phi}_2(1-\epsilon, x_2''') = 0$. This implies, by $x_2''' \geq 1 - 2\epsilon$,

$$\frac{\partial p_1}{\partial x_2}(1-\epsilon, x_2''') > 2\frac{1-x_2'''}{1+x_2'''}.$$

Then, $(1-\epsilon, x_2''')$ is asymptotically stable. Thus, the selfish equilibrium $(1,1)$ is asymptotically stable under any assortative matching rule. □

Intuitively, fair responders are easy to encounter fair proposers than selfish responders by assortativity. Hence, if proposers are almost fair ($x_1 \approx 0$), then strategy N generates higher average payoff than strategy Y. If proposers are almost selfish ($x_1 \approx 1$), in contrast, an average utility of strategy N is smaller than an average payoff of strategy Y. Therefore, the fully fair equilibrium and the selfish equilibrium coexist.

[4] A counter-example is given in section 4.

4 An Example

In this section, we give an example of a completely assortative matching rule under which only the fully fair equilibrium and the selfish equilibrium are asymptotically stable.

Definition 5. A matching rule $(p_i, q_i)_{i=1,2}$ is called *completely assortative* if the matching probability in population 1 is defined as

$$p_1 = \begin{cases} \frac{x_2}{x_1} & \text{if } x_1 > x_2 \\ 1 & \text{otherwise} \end{cases}$$

$$q_1 = \begin{cases} \frac{1-x_2}{1-x_1} & \text{if } x_1 \leq x_2 \\ 1 & \text{otherwise,} \end{cases}$$

and the matching probability in population 2 is defined as

$$p_2 = \begin{cases} \frac{x_1}{x_2} & \text{if } x_1 \leq x_2 \\ 1 & \text{otherwise} \end{cases}$$

$$q_2 = \begin{cases} \frac{1-x_1}{1-x_2} & \text{if } x_1 > x_2 \\ 1 & \text{otherwise.} \end{cases}$$

Figure 3 shows the probability with which selfish proposers may meet selfish responders under this assortative matching rule and under the random matching rule. As can be seen in Figure 3, fair proposers meet more likely fair responders than selfish proposers. The completely assortative matching rule maximizes a number of pairs which consist of a fair proposer and a fair responder.

Under the completely assortative matching rule, the selection dynamics is described as follows:

case 1: $x_1 > x_2$

$$\dot{x_1} = g_1(x_1, x_2) = (x_1 - \epsilon)(1 - \epsilon - x_1)(3\frac{x_2}{x_1} - 2) \tag{9}$$

$$\dot{x_2} = g_2(x_1, x_2) = (x_2 - \epsilon)(1 - \epsilon - x_2)(1 - 2\frac{1 - x_1}{1 - x_2}). \tag{10}$$

case 2: $x_1 \leq x_2$

$$\dot{x_1} = g_1(x_1, x_2) = (x_1 - \epsilon)(1 - \epsilon - x_1)(3 - 2) \tag{11}$$

$$\dot{x_2} = g_2(x_1, x_2) = (x_2 - \epsilon)(1 - \epsilon - x_2)(\frac{x_1}{x_2} + 2 - 2\frac{x_1}{x_2} - 2). \tag{12}$$

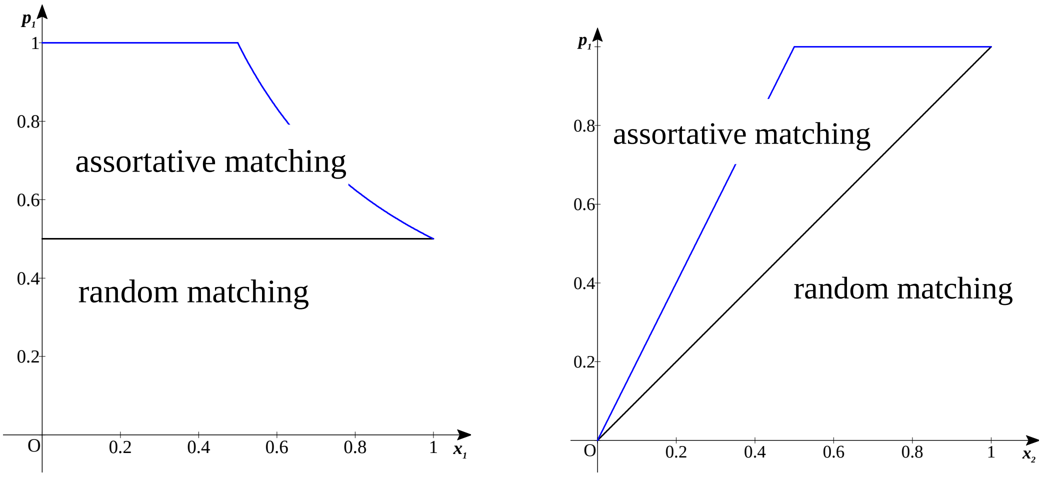

Figure 3: The probability $p_1 = Pr(L \text{ meets } Y)$. The graph in left side is $p_1(x_1, 0.5)$. The graph in right side is $p_1(0.5, x_2)$.

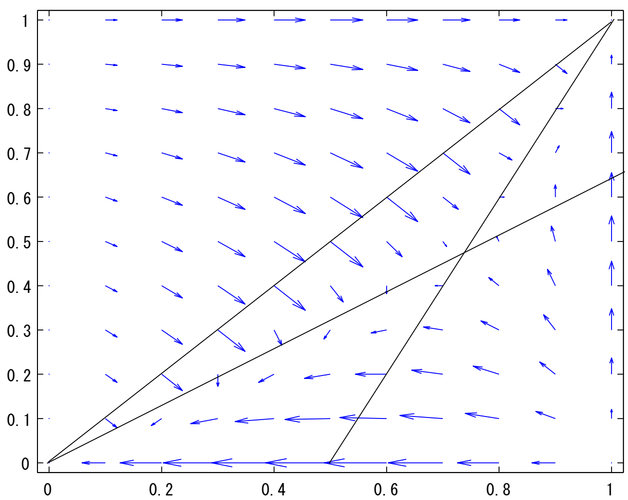

Figure 4: Phase diagram under the completely assortative matching rule.

This selection dynamics is a nonlinear dynamic system. Figure 4 shows the phase diagram of the system (9)-(12).

Proposition 4. *Let $\bar{\mathcal{A}}$ be the set of asymptotically stable points of the system (9)-(12). $\bar{\mathcal{A}} = \{(\frac{3}{2}\epsilon, \epsilon), (1-\epsilon, 1-2\epsilon)\}$.*

Proof. It is straightforward to see that the system (9)-(12) has the following set of rest points,

$$\bar{\mathcal{R}} = \{(\epsilon, \epsilon), (\epsilon, 1-\epsilon), (1-\epsilon, \epsilon), (1-\epsilon, 1-\epsilon)\} \\ \cup\{(\frac{3}{2}\epsilon, \epsilon), (1-\epsilon, 1-2\epsilon)(3/4, 1/2)\}.$$

To prove the proposition, we examine the eigenvalues of the Jacobian matrix. In the case of $x = (1-\epsilon, \epsilon)$, the Jacobian matrix at $(1-\epsilon, \epsilon)$ is

$$\frac{\partial g}{\partial x}(1-\epsilon, \epsilon) = \begin{pmatrix} (\frac{3\epsilon}{1-\epsilon} - 2)(2\epsilon - 1) & 0 \\ 0 & (1 - \frac{2\epsilon}{1-\epsilon})(1-2\epsilon) \end{pmatrix}.$$

Therefore, $(1-\epsilon, \epsilon)$ is not asymptotically stable for sufficiently small ϵ. Similarly, we can show that $x = (\epsilon, 1-\epsilon)$ is not asymptotically stable, either.

In the case of $x = (1-\epsilon, 1-\epsilon)$, it is not sufficient to consider the system in only one case, since two case of the system is surely included the neighborhood of $(1-\epsilon, 1-\epsilon)$. The both Jacobian at $(1-\epsilon, 1-\epsilon)$ in cases 1 and 2 are

$$\frac{\partial g}{\partial x}(1-\epsilon, 1-\epsilon) = \begin{pmatrix} 2\epsilon - 1 & 0 \\ 0 & 1-2\epsilon \end{pmatrix}.$$

Therefore, $(1-\epsilon, 1-\epsilon)$ is a saddle point for sufficiently small ϵ. We can show that $x = (3/4, 1/2), (\epsilon, \epsilon)$ is not asymptotically stable but a saddle point in the same manner.

Finally, we examine the case of $x = (\frac{3}{2}\epsilon, \epsilon)$. The Jacobian at $(\frac{3}{2}\epsilon, \epsilon)$ is

$$\frac{\partial g}{\partial x}(\frac{3}{2}\epsilon, \epsilon) = \begin{pmatrix} -\frac{2}{3} + \frac{5}{3}\epsilon & 1 - \frac{5}{2}\epsilon \\ 0 & \frac{-1+2\epsilon}{1-\epsilon} \end{pmatrix}.$$

Thus, $(\frac{3}{2}\epsilon, \epsilon)$ is asymptotically stable for sufficiently small ϵ. We can show that $(1-\epsilon, 1-2\epsilon)$ is asymptotically stable in the same manner. □

Proposition 4 shows that only the selfish equilibrium and the fully fair equilibrium survive in the replicator dynamics with the completely assortative matching rule.

5 Discussion

In this paper, we have studied the role of matching rules in the replicator dynamics in ultimatum mini game. If encounters are random, then Gale et al. [1995] shows that the subgame perfect equilibrium is the only one asymptotically stable point. However, many experimental observations do not support this result.

There are several possible explanations for fair actions. One explanation is inequity aversion. Subjects' preferences depend not only on their own monetary payoffs but on fairness or equity (Bolton and Ockenfels [2000], Fehr and Schmidt [1999]).

Another explanation is the reputation effect. In repeated situation, people worry about bad reputations and would act fairly (Nowak and Sigmund [1998], Nowak et al. [2000], Ohtsuki and Iwasa [2004]). If responders accept any unfair offer, this may become known and the next proposer will make unfair offers. To act fairly improves their long-term payoffs even if their short-term payoffs decrease.

Here we have considered the evolution of fair actions. An assortative matching rule (Becker [1973, 1974], Shimer and Smith [2000], Atakan [2006]) can be regarded as an alternative explanation of fair actions. The assortative matching rule make replicator dynamics to be a nonlinear system, and thus expands the set of stable points. For the ultimatum mini game, there exist some assortative matching rule supporting imperfect Nash equilibria. The average payoff of fair actions may become higher than selfish actions depending on the mass of fair agents. Therefore, the fair actions may survive. These assortative matching rules support fair actions as asymptotically stable states without fair preference or reputation.

Our study has some limitations. First, the selfish equilibrium is also asymptotically stable. The dynamic path depends on an initial state and an assortative matching. Second, it is not known that the same result holds for other types of dynamics. The ultimatum "mini" game is surely restricted. The analysis of general games in left for future works.

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