The Bidding Strategy in Proxied Package Auctions with Complementarities*

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Abstract

This paper investigates the bidding strategy in a package auction under incomplete information. I consider a simplified and limited case, where each bidder wants a unique bundle of the goods and evaluates them as perfect complements. The rule of the auction is a standard ascending auction with package bidding but I adopt the "proxy bidding rule." The auction is interpretted as a limited version of Ausubel and Milgrom (2002)'s ascending proxy auction. Moreover, in some cases the rule of the auction also covers the combinatorial clock auction by Porter et al (2003). I derive the condition under which bidders report their values truthfully in a Bayesian Nash equilibrium even when there're complementarities. Truthful reporting is optimal when bidders' geographic relationships about thier wants are transitive. This result implies that package auctions cannot implement efficient outcome in almost every case of package auction problems.

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1 Introduction

Since the design of the auction of spectrum licenses by Federal Communication Commission, it is an important problem how to sell the multiple items those values are interrelated. Some auction designs allow bidders to submit bids for packages of

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goods.[1] Particularly, when there may be high complementarities among the goods, it is said that auctions with package bidding will improve the efficiency.

Recent studies show that goods complementarities causes difficulty in both theory and applications.[2] From theoretical point of view, when goods may be complements, often there is no mechanism that satisfies desired properties. Many familiar package auctions are known not to satisfy incentive compatibility. However, because of its complexity of rules and the vast strategy space, few studies have so far been made at incentive structure of package auctions.

This paper formulates a model of a package auction, which is very simplified and limited but incomplete information model with complementarities. And I limit bidders' strategy space by adopting a "proxy auction" rule and investigate bidders' incentive properties in an ascending auction with package bidding.

I consider the case where each bidder evaluates goods as perfect complements. That is, each bidder is interested in a unique bundle of goods and makes profits only when he obtains all of the goods in that bundle. For example, in a spectrum auction, each mobile phone company has his business areas, and wants all of licenses covering those areas. There will be no use for licenses out of his areas and for only a part of licenses in the area.

The allocation of the goods is determined by an ascending package auction. And I adopt the "proxy bidding rule." In a proxy auction, bidders report their valuations on goods to their agents. Agents participate in an ascending auction and raise bids automatically up to maximum bids reported by bidders.[3] Proxy auctions are said to be more effective against dishonest behavior such as collusions and shill biddings than non-proxy auctions.[4] Moreover, the proxy rule accelerates the implementation of the auction (Parkes (2006)). Proxy rule restricts strategy space and makes it easy to analyze strategic behavior.

[1] Milgrom (2004) explains the FCC auctions and the basic theory of single- and multi-object auctions in detail. Parkes (2006) surveys theoretical and computational problems of iterative combinatorial auctions.

[2] In real world problems, goods complementarities matter well. See Milgrom (2007) for the difficulties in designing package auctions. Ausubel et al (1997) show the evidence of synergies in a FCC spectrum auction.

[3] A general model of a proxy auction with package bidding is introduced by Ausubel and Milgrom (2002). The model of this paper is a special case of their "ascending proxy auction." See Ausubel and Milgrom (2002, 2006).

[4] The rich information structure and the vast strategy space of dynamic (ascending) auction can make it possible for bidders to collude with each other and to lower prices. Brusco and Lopomo (2002) show there's a collusive Bayesian Nash equilibrium in an ascending multi-object auction without package bidding.

I show that bidders' relationships about their wants, in other words, bidders' geographical relationships, take an important role in bidding strategy in an equilibrium. When some two bidders are interested in a same good, then they are *rivals* of each other. The necessary and sufficient condition for each bidder reporting his willingness to pay truthfully is the transitivity about these rival relations, i.e. any rival of rival must be a rival. When this is not satisfied, a bidder has a *friend* who competes against a common rival. In package auctions, friends should cooperate in order to outbid the common rival in general. However, a kind of a freerider problem about their shares of payment arises in such a case. Bidders will underbid their values to avoid large shares of the payment.

This freerider problem has been pointed as the "threshold problem" in the literature of package auction theory.[5] The threshold problem is a coordination problem between local bidders (friends). Local bidders can outbid global bidders (common rivals) by coordinating their bids, but incentive to avoid large payment results in a coordination failure and they may not win the auction. This paper analyzes this problem under incomplete information and show that it's a main problem of package auctions.

One of the main contributions is that I formulate a limited model of package auctions and analyze bidders' incentive properties. In addition, I clarify the main weak point of package auctions partially by the simplified model. I show that the freerider problem in package auctions arises almost necessarily.

I start with viewing the limited model and the procedure of the auction using an example in section 2. Section 3 provides the model and introduces the protocol formally. Section 4 analyzes incentive structure in the simplest case described in section 2. In section 5 I formally define the "transitivity condition" and provide main results.

2 The Procedure of the Auction

I start by viewing how the ascending auction procedes before defining the model formally. Consider that there're two objects, $\{A, B\}$, and three bidders, $\{1, 2, 3\}$. Bidder 1 wants only good A, while bidder 2 wants only good B. Bidder 3, on the other hand, wants both A and B and values zero for either of them alone. These circumstances are illustrated in figure 1.

[5] See Ledyard et al (1997) and Milgrom (2000) for example.

In the auction, firstly bidders report their maximal bids to their agents, or "proxies." Then the proxies participate in an ascending auction submit bids up to the reported maximum values.

Suppose that bidder 1 and 2 report values of 4 and 2, respectively, to their proxies. And suppose that bidder 3 reports 4 for the package of the goods, $\{A, B\}$.

Figure 2 illustrates the process of the ascending auction. In period 1, each (proxied) bidder submit 1 for the (set of) items. In each period, the auctioneer selects a feasible revenue-maximizing bids and chooses provisional winners. In period 1, when the seller allocates the goods for bidder 1 and 2, its revenue is 2. On the other hand, if she allocates them to bidder 3, the revenue is only 1. So the auctioneer chooses the allocation $(\{A\}, \{B\}, \emptyset)$ and bidder 1 and 2 are provisional winners.

In period 2, only bidder 3 raises the bid to 2. Then $(\emptyset, \emptyset, \{A, B\})$ is also a revenue-maximizing allocation, whose revenue is 2. Suppose the seller chooses the allocation in which bidder 3 wins.

In period 3, bidder 1 and 2 raise their bids and submit 2. The auctioneer allocates the goods to bidder 1 and 2. Now the revenue is 4. In period 4 and 5, bidder 3 raise the bid up to 4. Suppose that the auctioneer selects the bid of bidder 3 in period 5.

In period 6, bidder 1 and 2 are going to raise bids. However, bidder 2's bid has already reached to 2, the maximal bid. So bidder 2 cannot raise his bids any more and only bidder 1 raises the bid to 3. Then, bidder 3's bid is already binding at the maximal bid, 4, so the auction stops. In the final allcation, bidder 1 gets A at 3, bidder 2 gets B at 2, and bidder 3 gets nothing.

3 The Model and the Auction

A seller wants to allocate some heterogeneous and indivisible commodities among a set of bidders. Let $K = \{1, \ldots, k\}$ be the finite set of goods for sale, which are different from each other. Let $I = \{1, \ldots, n\}$ be the set of all bidders. The seller, or the auctioneer, is denoted by 0. All bidders and the seller are risk-neutral and have quasi-linear utilities. let $\mathcal{K}$ be the power set of K and let X_i be the set of bidder i's possible packages, i.e. $X_i \subseteq \mathcal{K}$. We assume $\emptyset \in X_i$ for all $i \in I$. $v_i : X_i \to \mathbb{R}_+$ denotes bidder i's value function. We normalize $v_i(\emptyset) = 0$ for all $i \in I$. Bidder i gets his payoff of $v_i(x_i) - p_i$ $(x_i \in X_i,)$ where p_i is the monetary transfer to the seller. $x \equiv (x_1, \ldots, x_n)$ denotes an allocation, and x is *feasible* if for $\forall i, j \in I$, $x_i \cap x_j = \emptyset$ and $\bigcup_{i \in I} x_i \subseteq K$. Let X be the set of feasible allocations. Seller's utility is the revenue from the trades, $\sum p_i$.

3.1 Bidders' Valuations

I impose some assumptions on bidders' value functions. This paper assumes that every bidder values the items as perfect complements. And I consider the case where each buyer's private information is described as one-dimensional value. Following assumptions can simplify the situation of complicated package auctions.

A1 (*Perfect Complements*) There exists a nonempty bundle of the goods $K_i \in X_i$ for each $i \in I$, which is the bidder i's *region.* And

$$v_i(x_i) = \begin{cases} v_i & \text{if } K_i \subseteq x_i \\ 0 & \text{otherwise} \end{cases}, \text{ where } v_i \geq 0.$$

A2 (*Known Regions*) Each bidder's region, K_i is publicly known to each other and the seller, while v_i is private information for bidder i.

A3 (*Independent Values*) Each bidder's value, v_i is drawn from some continuous distribution on the interval $[0, \bar{v}_i] \equiv V_i$ independently.

The key assumption is *Perfect Complements.* It means that each bidder is interested in a unique package of the goods. When K_i contains more than one item, it indicates that there're perfect complementarities across these goods. Given *Perfect Complements*, *Known Regions* is not too restrictive. Even if each K_i is also private information, if *Perfect Complements* is common knowledge and if the seller requests potential bidders to report their regions before the auction, bidders will report their regions truthfully. Given *Perfect Complements* and *Known Regions*, *Independent Values* is standard in the literature of the auction theory.

3.2 The Auction, Strategy, and the Equilibrium Concept

Next I introduce the rule of the package auction. In the auction, each bidder instructs a "proxy agent" that participates in a simultaneous ascending auction with package bidding on his behalf. Firstly each bidder inputs a value function. Then the agent for bidder i bids *straightforwardly* according to the reported valuation. In other words, proxy bidders behave as price-taker.

The proxy rule is adopted in many internet auctions, in which potential participants aren't always involved. A package auction with proxy bidding is introduced by

Ausubel and Milgrom (2002). The auction noted here is a simplified and limited version of the ascending proxy auction by Ausubel and Milgrom.[6] Ausubel-Milgrom's proxy auction is defined to have a discrete price increase rule and authers consider the case where the price increment is negligibly small. I also define the auction with discrete price increases first and then consider the case where the price increment converges to zero.

I impose another assumption on bidders' behavior in the auction in order to simplify the analysis. I consider the case where bidders report not their value function ($v_i(\cdot)$) but only their values for their regions, v_i. It forbid bidders to bid for any packages that they don't want.

A4 (*No Cross-Bids*) $X_i = \{\emptyset, K_i\}$ for all $i \in I$.[7]

Given these assumptions, the package auction is defined as follows.

1. Each bidder inputs the valuation for his region, $b_i \in V_i$.

2. At period 0, initialize $x_i^0 = \emptyset$ and $p_i^0 = 0$ for all $i \in I$.

3. At period t (≥ 1), the agent for bidder i bids p_i^t on the package K_i and

$$p_i^t = \begin{cases} p_i^{t-1} + \epsilon & \text{if } x_i^{t-1} = \emptyset \text{ and } p_i^{t-1} + \epsilon \leq b_i \\ p_i^{t-1} & \text{otherwise} \end{cases}$$

   where $\epsilon > 0$ is an exogenous bid increment.

4. The seller chooses a feasible allocation $x^t \in X$ that maximizes her revenue under the price vector $p^t = (p_i^t)_{i \in I}$. If there're more than one allocation that maximize the revenue, she selects one of them randomly.

5. Repeat 3-4, and if $p^T = p^{T-1}$, then the auction ends and x^T is the final allocation. Bidders such that $x_i^T = K_i$ win the package K_i with the price p_i^T.

I limit attention to the case in which the bid increment ϵ is negligibly small, and I consider the case of $\epsilon \to 0$ (and t increases continuously). In the appendix I provide a class of ascending auctions with continuous price increase, which includes

[6] The protocol of the ascending auction in the Ausubel-Milgorm auction is identical to *iBundle* by Parkes and Ungar (2000). So our rule of the ascending auction is also same as iBundle.

[7] It may veil some important part of strategic behavior in the package auction with high complementarities, *cross-bidding*. For example, when there're complementarities, increasing the price for the good B can reduce the demand for the good A. So the bidder who wants only the good A might be able to kick out the global bidders earlier by bidding for the good B.

the auction here. I describe the *mechanism* of the auction as (g, p). When the profile of bidders' reports is $b = (b_1, \ldots, b_n)$, the outcome is represented by $\big(g(b), p(b)\big) \in (X, \mathbb{R}^n)$, which specifies the choice of the allocation $x = g(b) \in X$ and the payment $p_i = p_i(b) \in \mathbb{R}$. The associated payoffs are given by $u_i(b; v_i) \equiv v_i(g_i(b)) - p_i(b)$ for $i \in I$ and $u_0(b) \equiv \sum p_i(b)$ for the seller.

Bidder i's *strategy* is a mapping from bidder's value to his bid: $\beta_i : V_i \to V_i$. I call β_i bidding function.[8]

I use the Bayesian Nash equilibrium as the equilibrium concept. It is convenient to define bidder's interim expected payoff given some other bidders' strategies: $\pi_i(b_i, v_i) \equiv E[v_i(g_i(b_i, \beta_{-i})) - p_i(b_i, \beta_{-i})|\beta_{-i}]$.

Definition. A profile of bidding functions $\{\beta_i\}_{i \in I}$ is a *Bayesian Nash equilibrium* if

$$\forall i \in I, \forall v_i \in V_i, \forall b_i \in V_i, \quad \pi_i(\beta_i(v_i), v_i) \geq \pi_i(b_i, v_i).$$

To rule out some unreasonable equilibria that consist of (weakly) dominated strategies, I only consider the equilibrium in which no strategy is dominated.

Definition. A strategy β_i is (weakly) dominated if $\exists \hat{\beta}_i : V_i \to V_i$, $\forall v_i \in V_i$, $\forall v_{-i} \in V_{-i}$,

$$u((\beta_i(v_i), v_{-i}); v_i) \leq u((\hat{\beta}_i(v_i), v_{-i}); v_i),$$

and for some (v_i, v_{-i}) strict inequality holds.

3.2.1 Basic Properties of the Auction

The model and the rule of the auction is a simplified version of Ausubel and Milgrom (2002)'s "ascending proxy auction." Ausubel and Milgrom (2002, 2006) formulate a general model of the package auction problem and investigate incentive under complete information and derive Nash equilibrium.[9] Their results holds for the package auction here too.

The crucial fact used here is that the auction computes an efficient allocation with respect to the reported preferences. When I consider the efficiency of the auction, it is convenient to define the total value of the auction. Ausubel and Milgrom define and use coalitional value function for treating the auction as a cooperative game. In

[8] I consider only pure strategies and don't think of mixed strategies.

[9] Ausubel and Milgrom show that if goods are substitutes for all bidders, truthful reporting, $\beta_i(v_i) = v_i$, is a weakly dominant strategy of the ascending proxy auction.

my setting of incomplete information the total value depends on private information explicitly.

Definition. Coalitional value function $w : 2^I \times \prod V_i \to \mathbb{R}_+$,

$$w(J, v) = \max_{x \in X} \sum_{j \in J} v_j \mathbf{1}_{\{x_j \supseteq K_j\}}.$$

$w(J, v)$ denotes the total value which every subset of bidders $J \subseteq I$ can achieve by allocating goods efficiently among them. Let $w(\emptyset, v) \equiv 0$. By definition, $w(J, v)$ does not depend on $(v_i)_{i \notin J}$, so that I often use the same notation $w(J, v)$ as $w\big(J, \big((v_j)_{j \in J}, (\tilde{v}_i)_{i \notin J}\big)\big)$ for arbitrary $(\tilde{v}_i)_{i \notin J}$, or sometimes I drop some of $(\tilde{v}_i)_{i \notin J}$ for simple description.

Let $v = (v_1, \ldots, v_n)$ be a profile of bidders' values. And let $u_i(v; v_i)$ be bidder i's payoff when bidder i's bid is equal to his valuation for his region. Ausubel and Milgrom show that the payoff profile $u(v) \equiv (u_0(v), u_1(v; v_1), \ldots, u_n(v; v_n))$ is in the core of the economy.

Theorem. (Ausubel and Milgrom (2002, 2006)) The payoff profile determined by the auction is a core with respect to the reported preferences:

$$\begin{aligned} u(v) &\in Core(I \cup \{0\}, v) \\ &\equiv \Big\{ u \in \mathbb{R}_+^{n+1} \;\Big|\; w(I, v) = u_0 + \sum_{i \in I} u_i,\ w(J, v) \leq u_0 + \sum_{j \in J} u_j \text{ for } \forall J \subset I \Big\}. \end{aligned}$$

4 An Illustration

In this section I consider the 2-good and 3-bidder case again. Let $K = \{A, B\}$, $I = \{1, 2, 3\}$, $K_1 = \{A\}$, $K_2 = \{B\}$, and $K_3 = \{A, B\}$. This simple model gives intuition of main theorems in the next section.

In this case, we can specify the auction mechanism and describe the auction as a normal form game. Let b_i be the bidder i's reported value. Then the mechanism of the auction is specified as

$$g(b) = \begin{cases} (\{A\}, \{B\}, \emptyset) & \text{if } b_1 + b_2 \geq b_3 \\ (\emptyset, \emptyset, \{A, B\}) & \text{otherwise} \end{cases},$$ [10]

[10] Precisely if $b_1 + b_2 = b_3$, winner(s) are determined randomly from $\{1, 2\}$ and $\{3\}$. Winners will have to pay $p_i = b_i$. It isn't important how to deal with the tie-breaking here.

and

$$p(b)=\begin{cases}(\frac{1}{2}b_3,\frac{1}{2}b_3,0) & \text{if } \min\{b_1,b_2\}\geq\frac{1}{2}b_3\\ (b_1,b_3-b_1,0) & \text{if } 2b_1<b_3\leq b_1+b_2\\ (b_3-b_2,b_2,0) & \text{if } 2b_2<b_3\leq b_1+b_2\\ (0,0,b_1+b_2) & \text{otherwise}\end{cases}.$$

First I note that bidder 3 has a dominant strategy of truthtelling. We can check this easily by following argument. Bidder 3 wins if and only if his report is bigger than b_1+b_2 and the amount of payment is determined by the highest rejected bid, b_1+b_2. This rule is the same as the second-price auction. Hence truthful reporting is a weakly dominant strategy for bidder 3.

Next consider the strategy of bidder 1. Fix the (equilibrium) strategies of bidder 2 and 3; β_2 and β_3. Bidder 1's interim expected payoff given his true value v_1 and the reported value b_1 is

$$\begin{aligned}\pi_1(b_1,v_1)=v_1\Pr\{b_1>\beta_3(v_3)-\beta_2(v_2)\}-b_1\Pr\{2b_1<\beta_3<b_1+\beta_2\}\\ -E[\frac{1}{2}\beta_3\mathbf{1}_{\{2b_1\geq\beta_3,\ 2\beta_2\geq\beta_3\}}]-E[(\beta_3-\beta_2)\mathbf{1}_{\{2\beta_2<\beta_3<b_1+\beta_2\}}].\end{aligned}$$

By differentiating by b_1 and using $\beta_3(v_3)=v_3$, we have the first-order condition for maximization,

$$(v_1-b_1)\phi(b_1)-\Pr\{2b_1<v_3<b_1+\beta_2(v_2)\}\leq 0, \tag{1}$$

where equality holds when $b_1>0$. And

$$\phi(b_1)=\int_{v_2}f_2(v)f_3(b_1+\beta_2(v))\mathrm{d}v$$

and

$$\Pr\{2b_1<v_3<b_1+\beta_2(v_2)\}=\int_{b_1<\beta_2}\int_{2b_1}^{b_1+\beta_2}f_2(v)f_3(u)\mathrm{d}u\mathrm{d}v,$$

where f_i is the density function of v_i. We summerize and have the following proposition.

Proposition 1. Bidder $i(=1,2)$ submits the value under his true value, i.e. $\beta_i(v_i)<v_i$ if bidder $j=3-i$ submits non-zero bid with positive probability and if $\Pr\{2v_i<v_3<v_i+\beta_j(v_j)\}>0$. On the other hand, Bidder 3 has a weakly dominant strategy of truth telling.

Now I assume that bidder 1 and 2 are symmetric and that all values are drawn from uniform distributions. Let $V_1=V_2=[0,1]$, and $V_3=[0,a]$ where $a\geq 2$. And $F_1(v)=F_2(v)=v$ and $F_3(v)=v/a$.

Then FOC of bidder 1's maximization problem is

$$(v_1 - b_1)\int_{v_2} \frac{1}{a}\mathrm{d}v - \int_{\beta_2 > b_1}\int_{2b_1}^{b_1+\beta_2} \frac{1}{a}\mathrm{d}u\mathrm{d}v \leq 0,$$

$$\therefore \quad (v_1 - b_1) - \int_{\beta_2(v)>b_1} \{\beta_2(v) - b_1\}\mathrm{d}v \leq 0. \tag{2}$$

Consider the equilibrium such that $\beta_1(\cdot) = \beta_2(\cdot) = \beta(\cdot)$ and β is continuous. The equation (2) yields

$$v_1 - \beta(v_1) \leq \int_{v>v_1} \{\beta(v) - \beta(v_1)\}\mathrm{d}v, \tag{3}$$

where equality holds if $\beta(v_1) > 0$. Evaluating (3) at $v_1 = 1$ yields $\beta(1) = 1$. And after some calculations, we have

$$\beta(v) = \begin{cases} 0 & \text{if } v < \mathrm{e}^{-1} \\ 1 + \log v & \text{if } v \geq \mathrm{e}^{-1} \end{cases}. \tag{4}$$

We can check $\beta(v) < v$ when $v \in (0, 1)$.

4.1 The Threshold Problem

This 2-good and 3-bidder model is a basic example when we explain the threshold problem in package auctions. Table 1 explains the threshold problem in this case. Local bidder 1 and 2 can outbid global bidders and obtain goods if at least one of them bid actively ("bid more") in the auction. On the other hand, when both abstain active bidding and stop raising bids, goods are sold to a global bidder. When both 1 and 2 bid actively, both bidders pay a moderate price and will get the (net) payoffs of 2. When only either of them, say bidder 1, bids actively, he must pay a higher price, and then his payoff is 1. Bidder 2, on the other hand, will get goods at a lower price, so that his payoff is large.

There are multiple Nash equilibria in which either of local bidders bids actively and the other stops bidding (and a mixed strategy equilibrium). Hence local bidders may not win the goods unless they successfully coordinate their bids to outbid global bidders. This is known as the threshold problem.

Under incomplete information, this multiple equilibria of the coordination problem is solved by bidders' private information. Figure 3 illustrates the equilibrium bidding function. If a bidder is weak, or a low type, he doesn't submit bids actively and drops out early. On the other hand, a strong, high type bidder bids actively and submits almost true valuation.

In this numerical example, incentive of lower types to freeride is quite strong. Bidding up to the willingness to pay raises the probability of win and never generates negative profits unless bids exceed the true valuation. However, for a low type, the profit in cases where he wins by "working hard" is actually very small because he really has to work hard (pay the maximum bid he set.) Rather, raising the limit of the bid increases the payment in cases where he wins even if he doesn't work hard, and this cost is relatively high.

On the other hand, a high type bidder expects that the opposing friend is relatively weak. Then the friend's bid is lower than mine whenever he bids actively or not. In addition, the cost of losing by shading bids is higher for the higher tpye bidder. So the high type bidder has no incentive to rely on the friend's contribution.

The equilibrium strategy also generates a serious low revenue problem. The numerical example implies that there may be cases where package auctions fail. Particularly, local bidders bid zero if their values are less than $1/\mathrm{e} \fallingdotseq 0.37$. The seller gets zero revenue at the probability of $(1/\mathrm{e})^2 \fallingdotseq 0.14$.[11]

5 Main Results

This section provides main results. Results in the previous section imply that bidders' relationships about their targets strongly affect their bidding strategies. I clarify that the key concept is the transitivity condition about bidders' geographical relationships. Bidders must satisfy the transitivity condition in order to have a dominant strategy of truthful reporting.

5.1 The Transitivity Condition

First I introduce the transitivity condition. It is defined on each bidder. Considering bidders' strategic biddings, one important point is bidders' geographical location, or what each bidder wants. The transitivity condition requires that any rival of the bidder's rival should also be a rival.

Definition. Bidder i satisfies the *transitivity condition* if the following condition

[11] Ausubel and Milgrom (2002) argue that one of disadvantages of the Vickrey auction is its non-monotonic revenue and that the Vickrey auction sometimes generates low revenue. The example in this section implies that the package auction may also result in low revenue because of strategic underbidding.

holds:

$$\forall j, j' \in I\ (j \neq j')\ K_i \cap K_j \neq \emptyset \text{ and } K_j \cap K_{j'} \neq \emptyset \Rightarrow K_i \cap K_{j'} \neq \emptyset.$$

To describe bidders' geographical relation, we define the concepts of rivals and friends.

Definition. Bidder j is a *rival* of bidder i if $K_i \cap K_j \neq \emptyset$. R_i denotes the set of rivals of i. Bidder j is a *friend* of bidder i if $K_i \cap K_j = \emptyset$. $C_i = I_{-i} \setminus R_i$ denotes the set of friends of i.

By using the notation of R_i and C_i, we can restate the transitivity condition as the following form.

Lemma 1. Bidder i satisfies the transitivity condition if and only if for $\forall j \in R_i$ and $\forall j' \in C_i$, $K_j \cap K_{j'} = \emptyset$.
Proof. All proofs appear in Appendix.

Figure 4 illustrates the transitivity condition. In top three cases, bidder i satisfies the transitivity condition, and in the bottom case not. The top right case shows that bidder i satisfies the transitivity condition, while bidder j and j' don't. The top left case is the fundamental case where the transitivity condition holds.

The transitivity is almost necessary and sufficient condition for truthful reporting. The top left case of figure 4 implies that bidder i (and also j and j') should outbid others by himself. This is the source of truthful reporting.

The transitivity condition is rarely satisfied in the real auctions. We'll check this by considering simple spectrum license auctions. Consider the situation where a government plans to allocate licenses of radio spectrum of a single band to mobile phone companies. The license is divided in two by regions, west and east. Some firms (local firms) want a part of the licenses, while others (global firms) need both of them. This is described by figure 5(i). In this case, global firm g satisfes the transitivity, but local firms, l_1 and l_2 don't.

Figure 5(ii) illustrates another case. Now the government sells two bands and each of them separated with east and west. Four licenses are to be sold. Global firms need the licenses of both regions, while local firms will want both bands of a single region in order to operate rich variety of services. In such a case no one satisfies the transitivity condition, so that everyone will have incentive to underbid.

5.2 The Dominant Strategy

First I consider the dominant strategy. The transitivity condition is the very condition for the existence of a dominant strategy. The key observations are that when bidder i wins, all rivals of i, R_i, lose the auction, and that all other winners are in C_i. In addition, the auction leads to an efficient allocation. These imply that bidder i wins the auction if and only if $w(I, b) = b_i + w(C_i, b) \geq w(I_{-i}, b)$.[12] When the transitivity condition is satisfied, friends of bidder i are also friends of i's rivals. This means that friends of bidder i have no influence in determination of allocation of bidder i. Then bidder i must outbid his rivals by his bid only. Furthermore, when bidder i wins, his payment is determined by the highest value of his rivals, by the ascending auction algorithm. That is, nothing differs from a simple single-unit ascending auction.

Theorem 1. Suppose A1-A4. The truthfully reporting strategy, $\beta_i(v_i) = v_i$, is a weakly dominant strategy for bidder i if and only if bidder i satisfies the transitivity condition.

5.3 The Bayesian Nash Equilibrium Strategy

When bidders have no dominant strategy, how about the optimal strategy in the Bayesian Nash equilbrium? Roughly speaking, when a bidder has no dominant strategy, a kind of freerider problem necessarily arises and there exists incentive to underbid.

Unfortunately, it is hard to specify a closed form of (g, p) in general. However, we can obtain some properties which are necessary for specifying bidders' decision problems. So first I provide some observations about the auction mechanism.

Fix others' bid. If a bidder wins the goods with the price of $\hat{p}_i < b_i$, bidder i's payment must never increase by increasing his bid. As long as he submits the value over $\hat{p}_i$, he wins the goods with $\hat{p}_i$ because the algorithm always stops at $\hat{p}_i$. Hence we have following observation.

Observation 1. For each $b \in V$, if $g_i(b) = K_i$ and $p_i(b) = \hat{p}_i < b_i$, then $g_i(\tilde{b}_i, b_{-i}) = K_i$ and $p_i(\tilde{b}_i, b_{-i}) = \hat{p}_i$ for all $\tilde{b}_i > \hat{p}_i$.

[12] Precisely, we should consider the case of tie. Particularly, under the equilibrium strategies which we consider in this section, tie-breaking rule does affect the result when winning bids are zero. However, tie-breaking rule doesn't affect our main theorems.

Next I define the minimal required bid. Let β be a profile of bidding functions and let $W \equiv w(I_{-i}, \beta_{-i}(v_{-i})) - w(C_i, \beta_{-i}(v_{-i}))$.[13] W is a random variable generated from v_{-i}. Remenber bidder i wins the auction if and only if $b_i + w(C_i, b) \geq w(I_{-i}, b)$. Hence when other bidders' strategies are β, W represents the minimal bid for bidder i to win the auction. Moreover, if bidder i wins, his payment p_i must be above W. This fact is obtained from Ausubel and Milgrom (2002).[14] These observations play important roles to derive bidder's first order condition of the payoff maximization problem.

Observation 2. $g_i(b) = K_i$ if $b_i > W$. And $b_i \geq W$ if $g_i(b) = K_i$. Further $p_i(b) \geq W$ if $g_i(b) = K_i$.

One problem is whether the distribution of W has a density function. Following lemma assures that the distribution of W has no atom except the corner when I limit the class of strategies.

Lemma 2. The distribution of W has no atom except $W = 0$ if for each $j \in I_{-i}$, β_j is continuous and strictly increasing as long as $\beta_j(v_j) > 0$.

Suppose that each β_j is a candidate of the equilibrium bidding strategy that satisfies the conditions of lemma 2. Φ denotes the distribution function of W and let ϕ be the density function of W such that $\phi(0) \equiv \lim_{\delta \to +0} \frac{\Phi(\delta) - \Phi(0)}{\delta}$.

Now we have the interim expected payoff of bidder i given his private informaiton v_i and the report b_i:

$$\pi_i(b_i, v_i) = v_i \Phi(b_i) - E[p_i(b_i, \beta_{-i})]. \tag{5}$$

Following lemmas help to investigate how the Bayesian Nash equilibrium strategy is. The equilibrium strategy must be nondecreasing and bidders never make any choice of overvidding.

Lemma 3. Bidder i's equilibrium strategy β_i is weakly increasing.

Lemma 4. For every bidder i, any bidding strategy $\tilde{\beta}_i$ such that for $\exists v_i$, $\tilde{\beta}_i(v_i) > v_i$ is weakly dominated by the following strategy $\hat{\beta}_i$, where $\hat{\beta}_i(v) = \min\{\tilde{\beta}_i(v), v\}$.

[13] For precise notation, $w(I_{-i}, \beta_{-i})$ should be described as $w(I_{-i}, (b_i, \beta_{-i}))$. We use simply $w(I_{-i}, \beta_{-i})$ and $w(C_i, \beta_{-i})$ as long as no confusion rises.

[14] Actually, $w(I_{-i}, b) - w(C_i, b)$ is equal to the payment of the Vickrey auction (Vickrey-Clarke-Groves mechanism; Vickrey (1961), Clarke (1971), Groves (1973)). The Vickrey payoff is known as the maximum possible payoff in the core of the economy. Our auction also leads to a core payoff, so that the price should be over the Vickrey price.

Combining these lemmas and observations about the mechanism of the auction, I have the first-order condition for bidder's maximization problem.

Lemma 5. Suppose A1-A4. And suppose that for each $j \in I_{-i}$, β_j is continuous and strictly increasing as long as $\beta_j(v_j) > 0$. Then the first-order condition for bidder i's maximization is given by

$$(v_i - b_i)\phi(b_i) - \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) = b_i\} \leq 0,\text{[15]} \tag{6}$$

where equality holds when $b_i > 0$.

Lemma 5 is explained intuitively as follows. When bidder i raises the bid by one unit marginally, then the probability of winning rises marginally by $\phi(b_i)$. In the case of such a new winning he will get the payoff of $(v_i - b_i)$ because the minimal required bid is exactly b_i. On the other hand, raising the bid increases the amount of the payment by one unit in the cases in which he wins the package K_i and the payment is *binding* at b_i. So the marginal payoff of bidder i is represented as the left hand side of (6).

Truthful reporting, $b_i = v_i$ is never optimal when

$$\Pr\{g_i(v_i) = K_i \wedge p_i(v_i) = v_i\} > 0.$$

And generally this is satisfied when the transitivity condition is not satisfied and when v_i is sufficiently low. Lemma 6 shows that if bidder i doesn't satisfy the transitivity and if all other bidders behave "actively" to some extent, truthful strategy is suboptimal for some types.

Lemma 6. Suppose A1-A4. And suppose that bidder i does not satisfy the transitivity condition. If β is a Bayesian Nash equilibrium strategy profile, each of which is continuous and $\beta_j(\bar{v}_j) > 0$ for $\forall j \in I$, then β_i is not truthful strategy and there exists some $\alpha > 0$ and $\beta_i(v_i) < v_i$ for all $v_i \in (0, \alpha)$.

Lemma 6 claims that the transitivity condition is almost necessary condition for truthtelling in the Bayesian Nash equilibrium. However, it is not exactly necessary condition and a slight difference arises because of the following consideration. There may be an equilibrium in which some bidders always submit zero, because there're the cases in which even if some bidders submit zero, they can win the goods by grace of others' high contribution. Therefore even if the transitivity condition is not

[15] $\Pr\{A \wedge B\}$ denotes the probability of the events in which both A and B occur.

satisfied, there're the case in which truthful reporting is still optimal. Considering the possibility of such a "zero-bidder," we imagine that zero-bidders are replaced from the market. The following theorem is second main theorem. Proof is given as a corrollary of theorem 1 and lemma 6.

Theorem 2. Suppose A1-A4. Let β be a strategy profile, each of which is continuous. And let $\hat{I} \equiv \{j \in I | \beta_j(\bar{v}_j) > 0\}$. If β is a Bayesian Nash equilibrium and if i satisfies the transitivity on $\hat{I}$, then $\beta_i(v_i) = v_i$ for all $v_i \in V_i$. On the other hand, if β is an equilibrium and if i does not satisfy the transitivity on $\hat{I}$, then there exists some $\alpha > 0$ and $\beta_i(v_i) < v_i$ for all $v_i \in (0, \alpha)$.

As we've already seen, the transitivity condition is restrictive in practical applications. Theorem 2 implies that if the auction is ex ante efficient all bidders must satisfy the transitivity condition.[16] However, this will not be the case in practice generally. If there are some local buyers in the market, they will frequently violate the transitivity condition. In addition, theorem 2 shows a negative result in another meaning. One of the aims of package auction is to find an optimal bundle of the goods. When all bidders satisfy the transitivity condition, the market is separated and closed with rivals. This means that what the optimal packaging is obvious and that there's no need to determine the bundling *during* the process of the auction. In other words, our package auction is ex ante efficient only when we don't have to design a package auction.

6 Concluding Remarks

I have analyzed the bidders' strategy in a package auction. One of the contributions of this paper is that we formulate a Bayesian model of a package auction with complementarities. Preceding studies construct models of complete information and derive some Nash equilibria of package auctions (Ausubel and Milgrom (2002), Milgrom (2007)). I derived a necessary and sufficient condition for the truthful dominant strategy property. Moreover, it is also almost necessary and sufficient condition for the incentive compatibility in the Bayesian Nash equilibrium.

I showed that almost all local bidders have incentive of underbidding when there're high complementarities. This finding corresponds to the "threshold prob-

[16]Even in the cases where all bidders underbid in the equilibrium, the degree of underbidding differs among bidders and private information. So ex ante efficiency will not be achieved when some bidders underbid in equilibrium.

lem" in package auctions, a coordination problem among local bidders. I adopted an incomplete information model and partially clarified the environment where the threshold problem, which is rather a freerider problem under incomplete information, takes place in package auctions.

A simplified and limited case gives us bidders' incentive properties in package auctions more. However, our results will be adaptable even when more general values are permitted. In the general values case, bidders will underbid more for small packages of the goods, while they will bid more sincerely on the larger packages. This bidding strategy, truthtelling on the large packages and underbidding on the small packages, is not a complex, rather simple strategy. The results are also testable in laboratory experiments.

Some combinatorial auctions achieve high efficiency in experimental studies.[17] However, little attention has been given to the point why high efficiency is achieved in experiments, for there're few studies on theoretical and experimental analysis about bidders' bidding strategies in combinatorial auctions. Many experimental studies have paid attention only to the efficiency and seller's revenue of auctions with and without package bidding. The results will be useful for analyzing and understanding strategic bidding behavior in experiments and real auction data. The theoretical analysis on bidders' behaviors in combinatorial auctions accompanied with experimentation will be in need much more.[18]

A Appendix

In the appendix, first I note the rule of ascending auction with continuous price increase. Then I provide proofs of lemmas and theorems.

A.1 A Continuous Price Package Auction

In the rule of the ascending auction described in section 3 the price increment is $\epsilon > 0$. We considered the case where the increment is negligibly small. As I've already noted, we cannot specify the monetary transter function of the package

[17] There're few experimental studies on proxied package auctions such as the ascending proxy auction. Kazumori (2006) test the "Clock-Proxy Auction" by Ausubel, Cramton, and Milgrom (2006). See Porter et al (2003) and Cybernomics (2000) for the experimental studies on several other multi-object auctions.

[18] Of course experimental economics take very important roles in design of auctions or markets. I'm also going to carry out the experiments in order to examine the bidders' bidding strategies in combinatorial auctions. Roth (2002) notes the importance of each of theory, experimentation, and computation.

auction, p. So I define the auction with continuous prices by imposing some axioms that it should satisfies.

First I define the provisionally winning coalitions. A subset of bidders $J \subset I$ is a *provisionally winning coalition at* t if $\big((K_j)_{j\in J}, (\emptyset)_{j\notin J}\big) \in X$ and $\sum_{j\in J} p_j(t) = w(I, p(t))$. $PW(t)$ denotes the set of provisionally winning coalitions at t.

Auctions which satisfy following properties include the one we consider in this paper. An package auction determines a price path $p(t) = (p_i(t))_{i\in I}$ as follows.

1. Each bidder inputs the valuation for his region, $b_i \in V_i$.
2. At time $t = 0$, initialize $p_i(0) = 0$ for all $i \in I$.
3. A price path $p(t)$ $(t \geq 0)$ is determined as follows. If $\dot{p}(T) = 0$ at T, go to the next step.
   (a) For all $j \in \bigcap_{J\in PW(t)} J$, $\dot{p}_j(t) = 0$.
   (b) For all $j \notin \bigcap_{J\in PW(t)} J$, $\dot{p}_j(t) = q_j^t > 0$ if $p_j(t) < b_j$. And $\dot{p}_j(t) = 0$ if $p_j(t) \geq b_j$.
   (c) For all $J, J' \in PW(t)$, $\sum_J \dot{p}_j(t) = \sum_{J'} \dot{p}_{j'}(t)$.
   (d) (Competitiveness) Suppose for $J \in PW(t)$, $i, j \in J$, $p_i(t) < b_i$ and $p_j(t) < b_j$. $[\forall J' \in PW(t), [j \in J' \Rightarrow i \in J']] \Rightarrow \dot{p}_i(t) \leq \dot{p}_j(t)$.
4. $t = T$. If for some $J, J' \in PW(T)$, $J \subset J'$, then $\hat{PW}(T) \equiv PW(T) \setminus \{J\}$ and replace $PW(T) \equiv \hat{PW}(T)$.
5. The seller choose randomly $J \in PW(T)$ and they are winners. Each winner pays $p_j(T)$.

One of important characteristics is that we don't completely specify the speed of each bidders' price increase (step 3(b)). But the speeds is restricted by step 3(c) and (d). Step 3(c) requires that any candidate of winning allocation should offer the maximum revenue at any time. Step 3(d) requires that when a bidder competes with more people, then the price increases fast. Step 3(d) also implies following *symmetry* property.

symmetry: Suppose for $J \in PW(t)$, $i, j \in J$, $p_i(t) < b_i$ and $p_j(t) < b_j$. If for $\forall J' \in PW(t)$, $[i, j \in J']$ or $[i, j \notin J']$, then $\dot{p}_i(t) = \dot{p}_j(t)$.

A.2 Proofs

Proof of Lemma 1. Suppose that for $\exists j \in R_i$ and $\exists j' \in C_i$, $K_j \cap K_{j'} \neq \emptyset$. However, the transitivity for bidder i requires $j' \in R_i$, which is contradiction. Conversely, the violation of the transitivity implies for $\exists j \in R_i$ and $\exists j' \in R_j$, $j' \in C_i$. ■

Proof of Theorem 1. (Sufficiency.) Suppose that bidder i satisfies the transitivity condition. Then for each $j \in R_i$ and each $j' \in C_i$, $K_j \cap K_{j'} = \emptyset$. So by construction of coalitional value function, we have

$$w(I_{-i}, b) = w(R_i, b) + w(C_i, b).$$

Note that if bidder i wins the items, $w(I, b) = b_i + w(C_i, b)$. So bidder i wins if and only if

$$b_i + w(C_i, b) \geq w(R_i, b) + w(C_i, b).$$

Hence bidder i wins if and only if $b_i \geq w(R_i, b)$. (If $b_i = w(R_i, b)$, winners are determined randomly.)

All bidders in R_i lose the auction when bidder i wins, so that at final period T in the ascending auciton they bid their reported values: $p_j^T = b_j$ for all $j \in R_i$. Whenever $p_i^t < w(R_i, b)$, there necessarily exists some blocking allocation against one which bidder i wins, and then the auction never stops. Once bidder i bids $p_i^t = w(R_i, b)$, then no allocation can block bidder i after that period. Therefore bidder i's winning bid is $p_i^T = w(R_i, b)$.

So bidder i's payoff is

$$u_i(b; v_i) = \begin{cases} v_i - w(R_i, b) & \text{if } b_i \geq w(R_i, b) \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that $b_i = v_i$ is a weakly dominant strategy.

(Necessity.) Suppose that for $j \in R_i$ and $j' \in C_i$, $K_j \cap K_{j'} \neq \emptyset$. Bidder i wins the goods if and only if $b_i + w(C_i, b) \geq w(I_{-i}, b)$. Let b_{-i} be such that $b_{j'} > b_j > 0$ and $b_m = 0$ for all $m \in I \backslash \{i, j, j'\}$. Then $w(C_i, b) = w(I_{-i}, b) = b_{j'}$ and $w(R_i, b) = b_j > 0$. Then even if bidder i bids zero, i wins and its price is zero. On the other hand, if i bids any $b_i > 0$, then he must win with price of $p_i > 0$ because $w(R_i, b) > 0$. ■

Proof of Lemma 2. Suppose that each β_j is continuous and strictly increasing as long as $\beta_j > 0$. And suppose that $w(I_{-i}, \beta_{-i}) - w(C_i, \beta_{-i}) = w > 0$.

Let I_{-i}^* and C_i^* be sets of bidders, who win the goods when goods are allocated efficiently among I_{-i} and C_i, respectively. That is,

$$w(I_{-i}, \beta_{-i}) = w(I_{-i}^*, \beta_{-i}) = \sum_{j \in I_{-i}^*} \beta_j$$

and

$$w(C_i, \beta_{-i}) = w(C_i^*, \beta_{-i}) = \sum_{j \in C_i^*} \beta_j.$$

Then

$$\sum_{j \in I_{-i}^*} \beta_j - \sum_{j \in C_i^*} \beta_j = w. \tag{7}$$

Because $w > 0$, there exists some $\hat{j}$ such that $\hat{j} \in I_{-i}^*$ and $\hat{j} \notin C_i^*$. Hence equation (7) always binds as a linear equality constraint of $(\beta_j)_{j \in I_{-i}}$. The continuity and the strict monotonicity of β assure that the set of v_{-i} which satisfies (7) must have zero measure. Further the way of selection of (I_{-i}^*, C_i^*) is finite, so $\Phi(w)$ has no atom when $w > 0$. ■

Proof of Lemma 3. Let $\Pi_i(v_i) \equiv \pi_i(\beta_i(v_i), v_i)$. Then

$$\Pi_i(v_i) \geq \pi_i(\beta_i(\tilde{v}_i), v_i) = \Pi_i(\tilde{v}_i) + v_i\Phi(\beta_i(\tilde{v}_i)) - \tilde{v}_i\Phi(\beta_i(\tilde{v}_i)),$$

hence

$$\Pi_i(v_i) - \Pi_i(\tilde{v}_i) \geq (v_i - \tilde{v}_i)\Phi(\beta_i(\tilde{v}_i)).$$

Similarly

$$\Pi_i(v_i) - \Pi_i(\tilde{v}_i) \leq (v_i - \tilde{v}_i)\Phi(\beta_i(v_i)).$$

Therefore

$$(v_i - \tilde{v}_i)\Phi(\beta_i(v_i)) \geq (v_i - \tilde{v}_i)\Phi(\beta_i(\tilde{v}_i)).$$

If $v_i > \tilde{v}_i$, then

$$\Phi(\beta_i(v_i)) \geq \Phi(\beta_i(\tilde{v}_i)).$$

Therefore $\beta_i(v_i) \geq \beta_i(\tilde{v}_i)$. ■

Proof of Lemma 4. Consider any v_i such that $\tilde{\beta}_i(v_i) > v_i$. If bidder i wins with the price $p_i < v_i$, he would also win the goods by submitting v_i with the same price (observation 1). When we fix the other bidders' bids, the auction stops at p_i whenever bidder i submits any $b_i > p_i$. If bidder i wins with the price over v_i, then he gets negative payoff. If he reports v_i, he will get zero payoff in that case. ■

Proof of Lemma 5. First we rewrite the interim expected payment $E[p_i(b_i, \beta_{-i})]$ as follows,

$$E[p_i(b_i, \beta_{-i})] = E[p_i(b_i, \beta_{-i}) \cdot \mathbf{1}_{\{p_i(b_i,\beta_{-i})<b_i\}}] + b_i \Pr\{g_i(b_i) = K_i \wedge p_i(b_i, \beta_{-i}) = b_i\}. \tag{8}$$

Expectations and probabilities are taken conditionally on bidder i's some report b_i and others' strategies β_{-i}.

Now we consider the marginal expected payment when bidder i increases the value by $\delta > 0$. We describe $p_i(b_i, \beta_{-i})$ simply as $p_i(b_i)$.

$$\begin{aligned} &E[p_i(b_i + \delta)] - E[p_i(b_i)] \\ &\quad = \big[E[p_i(b_i + \delta) \cdot \mathbf{1}_{\{p_i(b_i+\delta)<b_i+\delta\}}] - E[p_i(b_i) \cdot \mathbf{1}_{\{p_i(b_i)<b_i\}}]\big] \\ &\quad + b_i\big[\Pr\{g_i(b_i + \delta) = K_i \wedge p_i(b_i + \delta) = b_i + \delta\} - \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) = b_i\}\big] \\ &\quad + \delta \Pr\{g_i(b_i + \delta) = K_i \wedge p_i(b_i + \delta) = b_i + \delta\}. \end{aligned} \tag{9}$$

By observation 1, if $g_i(b_i + \delta, b_{-i}) = K_i$ and $p_i(b_i + \delta, b_{-i}) < b_i$, then $g_i(b) = K_i$ and $p_i(b_i + \delta, b_{-i}) = p_i(b_i, b_{-i})$. So

$$E[p_i(b_i+\delta)\cdot\mathbf{1}_{\{p_i(b_i+\delta)<b_i+\delta\}}]-E[p_i(b_i)\cdot\mathbf{1}_{\{p_i(b_i)<b_i\}}] = E[p_i(b_i+\delta)\cdot\mathbf{1}_{\{b_i\leq p_i(b_i+\delta)<b_i+\delta\}}].$$

And

$$\begin{aligned}E[b_i \cdot \mathbf{1}_{\{b_i \le p_i(b_i+\delta) < b_i+\delta\}}] &\le E[p_i(b_i+\delta) \cdot \mathbf{1}_{\{b_i \le p_i(b_i+\delta) < b_i+\delta\}}] \\ &< E[(b_i+\delta) \cdot \mathbf{1}_{\{b_i \le p_i(b_i+\delta) < b_i+\delta\}}],\end{aligned}$$

hence

$$\begin{aligned}b_i \Pr\{b_i \le p_i(b_i+\delta) < b_i+\delta\} &\le E[p_i(b_i+\delta) \cdot \mathbf{1}_{\{b_i \le p_i(b_i+\delta) < b_i+\delta\}}] \\ &< (b_i+\delta) \Pr\{b_i \le p_i(b_i+\delta) < b_i+\delta\}.\end{aligned} \tag{10}$$

Next, observation 1 also implies

$$\Pr\{g_i(b_i) = K_i \wedge p_i(b_i) < b_i\} = \Pr\{g_i(b_i) = K_i \wedge p_i(b_i+\delta) < b_i\}.$$

In addition, observation 2 implies

$$\Pr\{g_i(b_i) = K_i \wedge p_i(b_i+\delta) < b_i\} = \Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) < b_i\}.$$

Therefore

$$\Pr\{g_i(b_i) = K_i \wedge p_i(b_i) < b_i\} = \Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) < b_i\}. \tag{11}$$

Then, we have

$$\begin{aligned}&\Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) = b_i+\delta\} - \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) = b_i\} \\ &= \Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) \le b_i+\delta\} - \Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) < b_i+\delta\} \\ &\quad - \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) \le b_i\} + \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) < b_i\} \\ &= \Phi(b_i+\delta) - \Phi(b_i) - \Pr\{b_i \le p_i(b_i+\delta) < b_i+\delta\} \\ &\quad - \Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) < b_i\} + \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) < b_i\} \\ &= \Phi(b_i+\delta) - \Phi(b_i) - \Pr\{b_i \le p_i(b_i+\delta) < b_i+\delta\}.\end{aligned} \tag{12}$$

Using these equations (9), (10) and (12), we have following inequality after some calculation,

$$\begin{aligned}&b_i(\Phi(b_i+\delta) - \Phi(b_i)) + \delta \Pr\{g_i(b_i+\delta) = K_i \wedge p_i(b_i+\delta) = b_i+\delta\} \\ &\le E[p_i(b_i+\delta, \beta_{-i})] - E[p_i(b_i, \beta_{-i})] \\ &< b_i(\Phi(b_i+\delta) - \Phi(b_i)) + \delta \Pr\{g_i(b_i+\delta) = K_i \wedge b_i \le p_i(b_i+\delta) \le b_i+\delta\}.\end{aligned} \tag{13}$$

Therefore we have

$$\begin{aligned}&(v_i - b_i)\frac{\Phi(b_i+\delta) - \Phi(b_i)}{\delta} - \Pr\{b_i \le p_i(b_i+\delta) \le b_i+\delta\} \\ &< \frac{\pi_i(b_i+\delta, v_i) - \pi_i(b_i, v_i)}{\delta} \\ &\le (v_i - b_i)\frac{\Phi(b_i+\delta) - \Phi(b_i)}{\delta} - \Pr\{p_i(b_i+\delta) = b_i+\delta\}.\end{aligned} \tag{14}$$

Similarly we have

$$
\begin{aligned}
&(v_i - b_i + \delta)\frac{\Phi(b_i+\delta)-\Phi(b_i)}{\delta} - \Pr\{b_i - \delta \le p_i(b_i) \le b_i\} \\
&< \frac{\pi_i(b_i, v_i) - \pi_i(b_i-\delta, v_i)}{\delta} \\
&\le (v_i - b_i + \delta)\frac{\Phi(b_i+\delta)-\Phi(b_i)}{\delta} - \Pr\{p_i(b_i) = b_i\}.
\end{aligned} \tag{15}
$$

By taking limit $\delta \to 0$, we have

$$
\frac{\partial}{\partial b_i}\pi_i(b_i, v_i) = (v_i - b_i)\Phi(b_i) - \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) = b_i\}. \tag{16}
$$

So the FOC for maximization is represented as

$$
(v_i - b_i)\Phi(b_i) - \Pr\{g_i(b_i) = K_i \wedge p_i(b_i) = b_i\} \le 0, \tag{17}
$$

where equality holds if $b_i > 0$. ■

Proof of Lemma 6. Suppose that β is an equilibrium strategy profile, each of which is continuous and $\beta_i(\bar{v}_i) > 0$. By lemma 3, each β_i is weakly increasing.

Further suppose that each β_i is strictly increasing as long as $\beta_i > 0$. Then by lemma 5, if $\beta_i > 0$, the FOC of maximization problem is satisfied by equality. The left hand side of the FOC is linear in v_i, so that if $v_i > \hat{v}_i$, $\beta_i(v_i) > \beta_i(\hat{v}_i)$. Therefore if all other bidders satisfy the partial strict monotonicity property in the equilibrium, then the optimal reaction also satisfies the partial strict monotonicity property. So we apply lemma 5 and what to show is for some $\alpha > 0$, $\Pr\{g_i(v_i) = K_i \wedge p_i(v_i) = v_i\} > 0$ for $v_i \in (0, \alpha)$.

Now suppose that for $j \in R_i$ and $j' \in C_i$, $K_j \cap K_{j'} \neq \emptyset$.

Suppose that all bidders except bidder i play equilibrium strategies, which are continuous and strictly increasing when $\beta > 0$. We exhibit an measurable space of message profiles except bidder i, $B_{-i}(v_i) \subset V_{-i} = \prod_{m \in I_{-i}} V_m$ such that bidder i wins with value binding. Each bidder's strategy is non-decreasing and no one overbids in the equilibrium by lemma 3 and 4. Therefore for any $\hat{b}_m > 0$, the probability of the event that bidder m's equilibrium bid β_m is in the interval $[0, \hat{b}_m)$ is positive. Therefore if we propose a message profile space of positive measure, then the area of true value profile, which maps into that message profile space, is also positive measure.

Let $\bar{\beta}_m \equiv \beta_m(\bar{v}_m)$ for $m \in I$, and let α be some positive value and $0 < \alpha < \min\{\bar{\beta}_{j'}, \frac{1}{2}\bar{\beta}_j\}$.

Let $B_{-i}(v_i)$ be the set of profile of bids, b_{-i}, which satisfies following conditions C1-C5:

C1 $b_{j'} \ge \alpha$,

C2 $b_m < \frac{v_i}{|R_i \setminus \{j\}|}$ for all $m \in R_i \setminus \{j\}$ (if $R_i \setminus \{j\} \neq \emptyset$,) C3 $b_m < \frac{\alpha - v_i}{|R_{j'} \setminus \{j\}|}$ for all $m \in R_{j'} \setminus \{j\}$ (if $R_{j'} \setminus \{j\} \neq \emptyset$,)

C4 $b_m < \frac{\bar{\beta}_j - 2\alpha}{|C_i \cap C_{j'}|}$ for all $m \in C_i \cap C_{j'}$ (if $C_i \cap C_{j'} \neq \emptyset$,)

C5 $2v_i + w(R_{j'} \setminus \{j\}, b) + w(C_i \cap C_{j'}, b) < b_j < v_i + b_{j'} + w(C_i \cap C_{j'}, b)$.

Let $v_i < \alpha$. C2, C3, and C4 imply $w(R_i \setminus \{j\}, b) < v_i$, $w(R_{j'} \setminus \{j\}, b) + v_i < \alpha$, and $v_i + \alpha + w(C_i \cap C_{j'}, b) < 2\alpha + w(C_i \cap C_{j'}, b) < \bar{\beta}_j$, respectively. C4 guarantees that b_j which satisfies C5 exists and that $\Pr\{v_{-i} | \beta_{-i}(v_{-i}) \in B_{-i}(v_i)\} > 0$. C5 implies that bidder j loses, and that bidder i and j' win the auction.

Now let $b_i = v_i$ and let period t in the ascending auction be the very period at which all bidders in $R_{j'} \setminus \{j\}$ drop out and lose the auction, i.e.

$$t \equiv \min\, s\ [p^s_{j'} \geq w(I \setminus \{j', j\}, b) - w(C_{j'}, p^s)].$$

If $R_{j'} \setminus \{j\} = \emptyset$, let $t = 0$. This means that bidder j is the only rival of bidder j' after period t.

At period t, bidder j is still standing. Bidder j' [19] outbids the rivals except bidder j by bidding at most $w(R_{j'} \setminus \{j\}, b)$. At that period, on the other hand, other winning bidders $\{i\} \cup (C_i \cap C_{j'})$ bid at most $v_i + w(C_i \cap C_{j'}, b)$. Because $v_i + w(C_i \cap C_{j'}, b) + w(R_i \setminus \{j\}, b) < b_j$ by C5, bidder j never drops out before t and still *stands*.

Let $\tilde{b}_m \equiv b_m - p^t_m$ for each $m \in I_{-i}$, $\tilde{v}_i \equiv v_i - p^t_i$, and $\tilde{w}(C_i \cap C_{j'}) \equiv w(C_i \cap C_{j'}, b) - w(C_i \cap C_{j'}, p^t)$. Then

$$\begin{aligned} \tilde{b}_j = b_j - p^t_j &\geq 2v_i + w(R_{j'} \setminus \{j\}, b) + w(C_i \cap C_{j'}, b) \\ &\qquad - p^t_i - p^t_{j'} - w(C_i \cap C_{j'}, p^t) \\ &\geq \tilde{v}_i + v_i + \tilde{w}(C_i \cap C_{j'}) \\ &\geq 2\tilde{v}_i + \tilde{w}(C_i \cap C_{j'}). \end{aligned} \tag{18}$$

The first inequality in the equation (18) follows from C5 and the fact $p^t_j \leq p^t_i + p^t_{j'} + w(C_i \cap C_{j'}, p^t)$ when $\epsilon \to 0$. This inequality is obtained from the rule of the auction (step 3(c) of the auction with continuous price increases.) When the price increments are negligible, any candidate of final allocation must always offer the revenue maximizing prices. So at period t, bidder i, j', and j stand, so the allocation in which i and j' obtain the goods and one in which j obtains the items offer the same price: $p^t_j + w(C_j, p^t) = p^t_i + p^t_{j'} + w(C_i \cap C_{j'}, p^t)$. The second inequality follows from the definition of $\tilde{v}_i$ and $\tilde{w}(C_i \cap C_{j'})$, and from $p^t_{j'} \leq w(R_{j'} \setminus \{j\}, b)$.

Consider the auction after period t as a new auction with reported profile $(\tilde{v}_i, \tilde{b}_{-i})$. Bidder j is the only rival of bidder j', so bidder j' raises the bid if and only if bidder j is the provisional winner. And at those periods bidder i is not a provisional winner too, so that bidder i also raises the bid as long as bidder i's bid does not bind. Hence

[19] Hereafter we don't distinguish players in the game "bidders" from "agents" for bidders. We call both *bidder*.

$\dot{p}_i \geq \dot{p}_{j'}$ if $\dot{p}_i > 0$ (step 3(d) of the auction with continuous price increases.) So if bidder i wins with the price of $\hat{v}_i + p_i^t < \tilde{v}_i + p_i^t = v_i$, then bidder j' should win at $\hat{b}_{j'} + p_{j'}^t \leq \hat{v}_i + p_{j'}^t$. However, this means $\hat{v}_i + \hat{b}_{j'} + \tilde{w}(C_i \cap C_{j'}) \geq \tilde{b}_j > 2\tilde{v}_i + \tilde{w}(C_i \cap C_{j'})$, which is contradiction. Therefore bidder i wins with the price of $\tilde{v}_i + p_i^t = v_i$.

We have $\Pr\{g_i(v_i) = K_i \wedge p_i(v_i) = v_i\} \geq \Pr\{v_i | \beta_{-i}(v_{-i}) \in B_{-i}(v_i)\} > 0$. ■

References

[1] Ausubel, L. M., P. Cramton, R. P. McAfee, and J. McMillan (1997) "Synergies in Wireless Telephony: Evidence from the Broadband PCS Auctions," *Journal of Economics and Management Strategy*, 6, 497-527.

[2] Ausubel, L. M., P. Cramton, and P. Milgrom (2006) "The Clock-Proxy Auction: A Practical Combinatorial Auction Design," in *Combinatorial Auctions*, eds. P. Cramton, Y. Shoam, and R. Steinberg, MIT Press, 115-138.

[3] Ausubel, L. M., and P. Milgrom (2002) "Ascending Auctions with Package Bidding," *Frontier of Theoretical Economics*, 1, 1-42.

[4] Ausubel, L. M., and P. Milgrom (2006) "Ascending Proxy Auctions," in *Combinatorial Auctions*, eds. P. Cramton, Y. Shoam, and R. Steinberg, MIT Press, 79-114.

[5] Brusco, S, and G. Lopomo (2002) "Collusion via Signalling in Simultaneous Ascending Bid Auctions with Heterogeneous Objects, with and without complementarities," *Review of Economic Studies*, 69, 407-436.

[6] Clarke, E. H. (1971) "Multipart Pricing of Public Goods," *Public Choice*, 11, 17-33.

[7] Cybernomics (2000) "An Experimental Comparison of the Simultaneous Multiple Round Auction and the CRA Combinatorial auction," available at www.fcc.gov/wtb/auctions/comin/98540191.pdf.

[8] Groves, T. (1973) "Incentives in Teams," *Econometrica*, 41, 617-631.

[9] Kazumori, E. (2006) "Auctions with Package Bidding: An Experimental Study."

[10] Ledyard, J. O., D. Porter, and A. Rangel (1997) "Experiments Testing Multi Object Allocation Mechanisms," *Journal of Economics and Management Strategy*, 6, 639-675.

[11] Milgrom, P. (2000) “Putting Auction Theory to Work: The Simultaneous Ascending Auction,” *Journal of Political Economy*, 108, 245-272.

[12] Milgrom, P. (2004) *Putting Auction Theory to Work*, Cambridge University Press.

[13] Milgrom, P. (2007) “Package Auctions and Exchanges,” *Econometrica*, 75, 935-965.

[14] Parkes, D. C. (2006) “Iterative Combinatorial Auctions,” in *Combinatorial Auctions*, eds. P. Cramton, Y. Shoam, and R. Steinberg, MIT Press, 41-77.

[15] Parkes, D. C., and L. H. Ungar (2000) “Iterative Combinatorial Auctions: Theory and Practice,” in *Proceedings of the 17th National Conference on Artificial Intelligence (AAAI-00)*, 74-81.

[16] Porter, D., S. Rassenti, A. Roopnarine, and V. Smith (2003) “Combinatorial Auction Design,” *Proceedings of the National Academy of Sciences*, 100, September, 11153-11157.

[17] Roth, A. (2002) “The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics,” *Econometrica*, 70, 1341-1378.

[18] Vickrey, W. (1961) “Counterspeculation, Auctions, and Competitive Sealed Tenders,” *Journal of Finance*, 16, 8-37.

Table 1: Threshold problem.

1\2	bid more	stop
bid more	(2, 2)	(1, 3)
stop	(3, 1)	(0, 0)

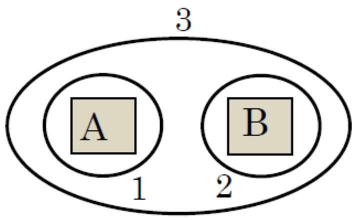

Figure 1: The 2-good and 3-bidder case.

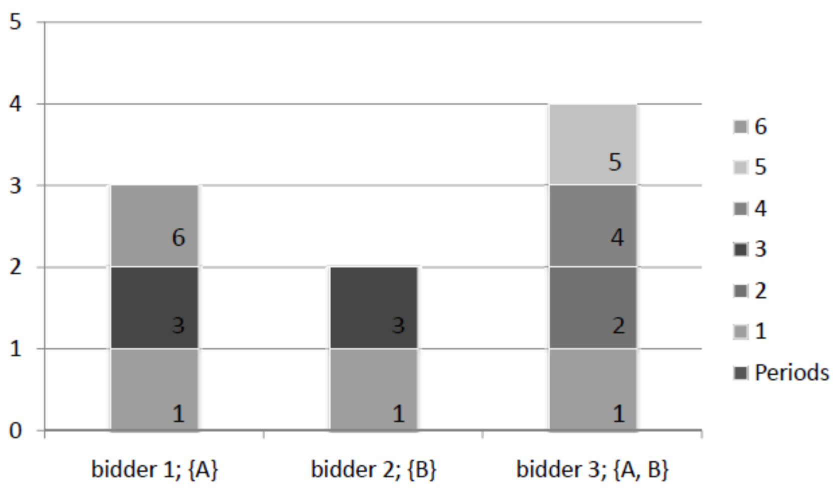

Figure 2: An example of the procedure of the auction.

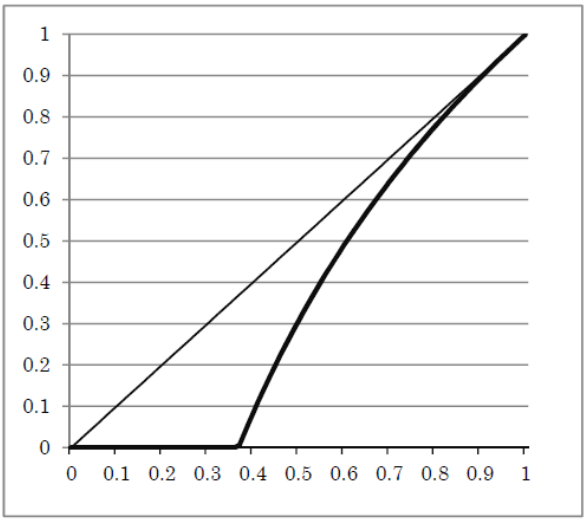

Figure 3: The equilibrium bidding function of bidder 1 and 2 in the 2-good and 3-bidder case.

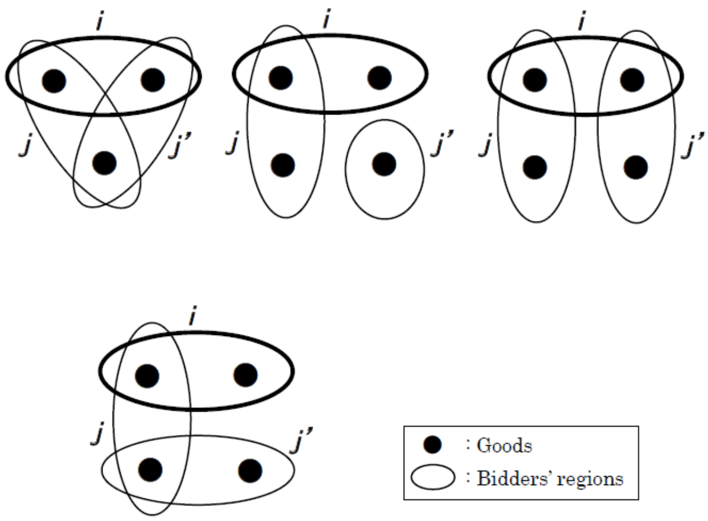

Figure 4: Bidder i satisfies the transitivity in top three cases, while not in the bottom case.

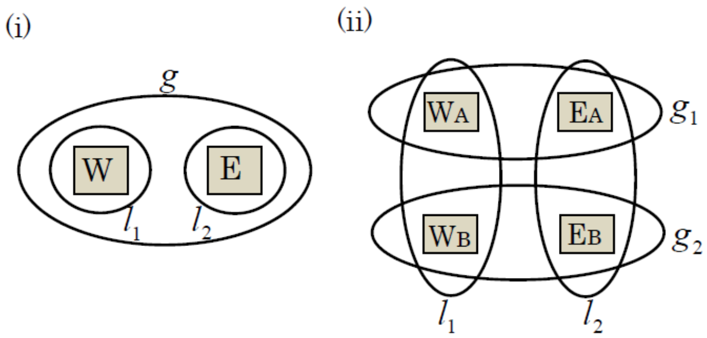

Figure 5: Two examples of spectrum auction.