Strategic complementarities and nested potential games*

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Abstract

This paper shows that every finite game of weak strategic complementarities is a nested pseudo-potential game if the action set of one player is multi-dimensional and the action sets of the others are one-dimensional; the implication does not hold, however, if the action sets of more than two players are multi-dimensional. Moreover, the paper proposes a new class of games of nested strategic complementarities that generalize the games of strategic complementarities and the nested potential games.

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Key words : strategic complementarities, nested potential games.

1 Introduction

This paper investigates a relationship between games of strategic complementarities and nested potential games, and proposes a new class of games that generalize those two classes of games.

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Games of strategic complementarities are games where, if her competitors turn more aggressive, an optimal reaction of a player is to become more aggressive as well. Many economic models belong to this class.[1] In the literature, various versions of strategic complementarities have been proposed and analyzed.[2] The weakest version of "strategic complementarities" is given by the weak strategic complementarities discussed in Dubey *et al.* (2006).

On the other hand, since Moderer and Shapley (1996), various versions of potential games also have been proposed and analyzed.[3] These potential games have in common an attractive feature that every maximizer of a potential, a real valued function over the set of action profiles, is a (pure) Nash equilibrium of the game. Uno (2007a) shows that the weakest version of potential games is the nested pseudo-potential games.

These two classes of games are useful at the following meanings: whenever we analyze a strategic situation as a game, we face the issue of the existence and identification of a (pure) Nash equilibrium. We can provide an affirmative answer to the existence question if the game has the structure of strategic complementarities or a potential function: if the game has strategic complementarities or a potential function, there exists a (pure strategy) Nash equilibrium in the game. If the game has strategic complementarities, a best response path from the greatest (least) action profile converges to an equilibrium; if the game has a potential function, a maximizer of the function is an equilibrium.

This paper shall show the followings: first, we will show that every finite game of strategic complementarities is a nested pseudo-potential game if the action sets of two or more players are multi-dimensional (Theorem 10). Next, we will illustrate that strategic complementalities do not necessarily imply the existence of a nested potential if the action

[1]See Topkis (1998) and Vives (1999) in detail.

[2]For example, games of strategic complementarities are including supermodular games introduced by Topiks (1979) and quasi-supermodular games introduced by Milgrom and Shannon (1994).

[3]For example, exact potentials, weighted potentials, ordinal potentials, generalized ordinal potentials are introduced in Monderer and Shapley (1996); (ordinal) best response potentials in Voorneveld (2000); pseudo-potentials in Dubey *et al.* (2006); best response potentials and better response potentials in Morris and Ui (2004); generalized potentials, monotone potentials, and local potentials in Morris and Ui (2005); iterated potentials in Oyama and Tercieux (2004); nested best response potentials in Uno (2007b); and so on.

sets of two or more players are multi-dimensional (Example 19). Finally, we will propose a new class of games which generalizes games of weak strategic complementarities and nested potential games (Definition 24), and show that there exists a pure strategy Nash equilibrium in such games (Corollary 25).

2 Preliminaries

Let X be a finite subset of m-dimensional Euclidean space. The inequality $x \geq y$ means $x_i \geq y_i$ for each i, while $x > y$ means $x \geq y$ and there exists i such that $x_i > y_i$.

For $x, y \in X$, let $\inf_X\{x, y\} := \sup\{z \in X | z \leq x, z \leq y\}$ denote the least upper bound for x and y in X, and let $\sup_X\{x, y\} := \inf\{z \in X | z \geq x, z \geq y\}$ denote the greatest lower bound for x and y in X.

A set X is a *lattice* if X contains the least upper bound and the greatest lower bound of each pair of its elements, i.e., for each $x, y \in X$, $\inf_X\{x, y\} \in X$ and $\sup_X\{x, y\} \in X$.

Tarski (1955) showed that the collection of fixed points of an increasing function from a nonempty finite lattice into itself is a nonempty lattice, and he gave the form of the greatest fixed point and the least fixed point:[4]

Theorem 1 (Tarski, 1955) *Suppose that f is an increasing function from a nonempty finite lattice X to X. Then, the set of fixed points of f in X is nonempty,* $\sup\{x \in X | x \leq f(x)\}$ *is the greatest fixed point, and* $\inf\{x \in X | x \geq f(x)\}$ *is the least fixed point.*

3 Strategic Complementarities

A strategic form game consists of a finite player set $N = \{1, \ldots, n\}$, an action set A_i for $i \in N$, and the payoff function $g_i : A \to \mathbb{R}$ for $i \in N$, where $A := \prod_{i \in N} A_i$. Since we fix the set A of action profiles, we denote a strategic form game $(N, (A_i)_{i \in N}, (g_i)_{i \in N})$ simply by $\mathbf{g}^N := (g_i)_{i \in N}$. For notational convenience, we write $a = (a_i)_{i \in N} \in A$; for $i \in N$,

[4] In fact, Tarski (1955) provides the fixed point theorem of an increasing function on a complete lattice instead of a finite lattice.

$A_{-i} = \prod_{j \neq i} A_j$ and $a_{-i} = (a_j)_{j \neq i} \in A_{-i}$; and for $T \subseteq N$, $A_T = \prod_{i \in T} A_i$, $a_T = (a_i)_{i \in T} \in A_T$, $A_{-T} = \prod_{i \in N \setminus T} A_i$, and $a_{-T} = (a_i)_{i \in N \setminus T} \in A_{-T}$. For each $T \subseteq N$, for any $a_{-T} \in A_{-T}$, let $\mathbf{g}^N|_{a_{-T}}$ denote the game where the action profile of players outside T is fixed to a_{-T}.

Since Topkis (1979), various notions of strategic complementarities have been introduced.[5] Among them, the weakest notion is the game of weak strategic complementarities. A game has weak strategic complementarities if, for each player, there exists a nondecreasing best-response selection:

Definition 2 A game $\mathbf{g}^N$ is a *finite game of weak strategic complementarities* if, for each $i \in N$, $A_i \subset \mathbb{R}^{m_i}$ is a finite lattice,[6] where $m_i \in \mathbb{N}$, and there exists a function $b_i : A_{-i} \to A_i$ such that

1. b_i is i's best-response selection: $b_i(a_{-i}) \in \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$, and

2. b_i is increasing with a_{-i}: $b_i(a_{-i}) \leq b_i(a'_{-i})$ whenever $a_{-i} < a'_{-i}$.

4 Nested Potential Games

Let $\mathbf{g}^N$ be a strategic form game. Beginning with Monderer and Shapley (1996), various notions of potential games have been proposed. Among them, one of the weakest notions is the nested pseudo-potential games introduced in Uno (2007a). To introduce the nested pseudo-potential games, we introduce the pseudo-potential games proposed by Dubey *et al.*(2006). A pseudo-potential of game $\mathbf{g}^N$ is a real valued function f on the set A of action profiles such that, for each player i, i's best-response against the other players' actions a_{-i} in the alternative game where i's payoff function is given by f is that in the original game $\mathbf{g}^N$:

[5] For example, the supermodular games introduced by Topkis (1979), the games of strategic complementarities introduced by Bulow *et al.* (1985), the quasi-supermodular game introduced by Milgrom and Shannon, and so on.

[6] We can also consider a version of games with compact action sets. In the version, it is difficult to show our main result hold, which we will discuss in Remark 20.

Definition 3 (Dubey *et al.*, 2006) A function $f : A \to \mathbb{R}$ is a *pseudo-potential* of $\mathbf{g}^N$ if, for each $i \in N$,

$$\arg\max_{a_i \in A_i} f(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i}) \tag{1}$$

for all $a_{-i} \in A_{-i}$. If $\mathbf{g}^N$ has a pseudo-potential, $\mathbf{g}^N$ is called a *pseudo-potential game*.[7]

We say that an action profile a^* is a *pseudo-potential maximizer* of $\mathbf{g}^N$ if $f(a^*) \geq f(a)$ for all $a \in A$.

Dubey *et al.* (2006) showed that a pseudo-potential maximizer, if it exists, is a Nash equilibrium of the underlying game:

Proposition 4 (Dubey *et al.*, 2006) *If $\mathbf{g}^N$ is a pseudo-potential game with a pseudo-potential maximizer a^*, then a^* is a Nash equilibrium of $\mathbf{g}^N$.*

If action sets are finite, every pseudo-potential game possesses a pure strategy Nash equilibrium, since there exists a maximizer of a function whose domain is finite.

Corollary 5 (Dubey *et al.*, 2006) *Every pseudo-potential game with finite action sets possesses a pure strategy Nash equilibrium.*

We shall extend Proposition 4 by introducing a weaker notion of potential where a 'pseudo-potential' is considered for each subset of players instead of the entire set. For a partition $\mathcal{T}$ of N, we define the partition $\mathcal{T}$ pseudo-potentials as follows:

Definition 6 (Uno, 2007a) Let $\mathcal{T}$ be a partition of N. A *partition $\mathcal{T}$ pseudo-potential* of $\mathbf{g}^N$ is a tuple $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$, where, for each $T \in \mathcal{T}$, $f_T : A \to \mathbb{R}$ satisfies that, for each $i \in T$,

$$\arg\max_{a_i \in A_i} f_T(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$$

[7] If the inclusion of (1) can be replaced by the equality, f is called an (ordinal) *best-response potential*, which is introduced in Voorneveld (2000). The pseudo-potentials generalize the (ordinal) best-response potentials. Morris and Ui (2004, 5) also introduced alternative best-response potentials, which are special classes of (ordinal) best-response potentials of Voorneveld (2000) and the pseudo-potentials in Dubey *et al.* (2006). See Morris and Ui (2004) for more discussion of this notion. We can apply the analogous arguments in this section to these best-response potentials of Morris and Ui (2004).

for all $a_{-i} \in A_{-i}$.

We denote such a partition $\mathcal{T}$ pseudo-potential $(\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ by $\mathbf{f}^{\mathcal{T}} := (f_T)_{T\in\mathcal{T}}$ since action sets $(A_T)_{T\in\mathcal{T}}$ can be derived from the partition $\mathcal{T}$ of N and the set A of action profiles in the original game $\mathbf{g}^N$.[8]

Notice that we can regard each $\mathcal{T}$-pseudo-potential $\mathbf{f}^{\mathcal{T}}$ as a strategic form game, where $\mathcal{T}$ is the player set; for each $T \in \mathcal{T}$, A_T is the action set of T; and for each $T \in \mathcal{T}$, f_T is the payoff function of T. The idea of the nested pseudo-potential games is to construct such games iteratively for a nested sequence of partitions:

Definition 7 (Uno, 2007a) A function $f : A \to \mathbb{R}$ is a *nested pseudo-potential* of $\mathbf{g}^N$ if there exist a positive integer K and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=1}^K = ((f_T^k)_{T\in\mathcal{T}^k})_{k=1}^K$ such that

- $\{\mathcal{T}^k\}_{k=1}^K$ is a nested sequence of N: $\{\mathcal{T}^k\}_{k=1}^K$ is an increasingly coarser sequence of partitions of N with $\mathcal{T}^1 = \{\{i\}|i \in N\}$ and $\mathcal{T}^K = \{N\}$;
- $\mathbf{f}^{\mathcal{T}^1} = (f_T^1)_{T\in\mathcal{T}^1}$ is the original game $\mathbf{g}^N$: for each $i \in N$, $f_{\{i\}}^1(a) = g_i(a)$ for all $a \in A$;
- for each $k = 2, 3, \dots, K$, $\mathbf{f}^{\mathcal{T}^k} = (f_T^k)_{T\in\mathcal{T}^k}$ is a $\mathcal{T}^k$-pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}} = (f_T^{k-1})_{T\in\mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above: for each $T^k \in \mathcal{T}^k$ and each $T^{k-1} \in \mathcal{T}^{k-1}$ with $T^{k-1} \subseteq T^k$,

  $$\arg\max_{a_{T^{k-1}} \in A_{T^{k-1}}} f_{T^k}^k(a_{T^{k-1}}, a_{-T^{k-1}}) \subseteq \arg\max_{a_{T^{k-1}} \in A_{T^{k-1}}} f_{T^{k-1}}^{k-1}(a_{T^{k-1}}, a_{-T^{k-1}})$$

  for all $a_{-T^{k-1}} \in A_{-T^{k-1}}$; and
- $\mathbf{f}^{\mathcal{T}^K} = (f_N^K)$ is such that $f_N^K(a) = f(a)$ for all $a \in A$.

A game that admits a nested pseudo-potential is called a *nested pseudo-potential game.*

[8]The partition $\mathcal{T}$ pseudo-potential generalizes Monderer (2007)'s q-potential: a strategic form game $\mathbf{g}^N$ has a q-potential if and only if $\mathbf{g}^N$ has a partition $\mathcal{T}$-potential, where q refers to the number of elements in $\mathcal{T}$ and the potential is meant to be the exact potential in Monderer and Shapley (1996). If $\mathbf{g}^N$ is a q-potential game, then it has a partition $\mathcal{T}$ pseudo-potential such that the number of elements of $\mathcal{T}$ is q. The converse is not true, since there is a pseudo-potential game without an exact potential.

We say that an action profile a^* is a *nested pseudo-potential maximizer* of $\mathbf{g}^N$ if $f(a^*) \geq f(a)$ for all $a \in A$.

The essential property shared by all existing versions of potential games is that maximizers of a potential function are Nash equilibria as in Proposition 4. The nested pseudo-potential proposed here inherits this property. Indeed, Uno (2007a) showed that a nested pseudo-potential maximizer, if it exists, is a Nash equilibrium of the underlying game:

Proposition 8 (Uno, 2007a) *Let $\mathbf{g}^N$ be a nested pseudo-potential games with a nested pseudo-potential maximizer a^*. Then, a^* is a Nash equilibrium of $\mathbf{g}^N$.*

Proposition 8 implies the following corollary.

Corollary 9 (Uno, 2007a) *Every nested potential game with finite action sets possesses a pure strategy Nash equilibrium.*

5 Nested Potentials in Games of Strategic Complementarities

This section shows that games of weak strategic complementarities are nested pseudo-potential games if the action set of one player is multi-dimensional and the action sets of the others are one-dimensional.[9]

Theorem 10 *Let $\mathbf{g}^N$ be a game of weak strategic complementarities, where $A_i \subset \mathbb{R}^m$ for some unique player $i \in N$ and for some $m \in \mathbb{N}$, and $A_j \subset \mathbb{R}$ for any $j \neq i$. Then $\mathbf{g}^N$ is a nested pseudo-potential game.*

To prove the above theorem, we will use the following four facts.

Firstly, a game of weak strategic complementarities has a property that, for each subset T of N, for any action $a_{-T} \in A_{-T}$ of all players outside T, the Nash equilibrium of the restriction $\mathbf{g}^N|_{a_{-T}}$ of $\mathbf{g}^N$ is an increasing with respect to a_{-T}:

[9] As shown in Example 19, a game with strategic complementarities is not necessarily a nested potential game when the action sets of more than one player is multi-dimensional.

Lemma 11 *Let $\mathbf{g}^N$ be a game of weak strategic complementarities. Let T be a subset of N. For any $a_{-T} \in A_{-T}$, let $\mathbf{g}^N|_{a_{-T}}$ be the restricted game by a_{-T}. Then, there exists a function $e_T : A_{-T} \to A_T$ such that*

1. *e_T is an equilibrium selection: $e_T(a_{-T})$ is a pure strategy Nash equilibrium of $\mathbf{g}^N|_{a_{-T}}$ for any $a_{-T} \in A_{-T}$; and*
2. *$e_T(a_{-T})$ is increasing with a_{-T}: $e_T(a_{-T}) \leq e_T(a'_{-T})$ whenever $a_{-T} < a'_{-T}$.*

This lemma resembles the result from monotone comparative statics where a function from a nonempty lattice into itself has an increasing fixed point with the parameter. The following proof is also similar to that of the monotone comparative statics in Milgrom and Roberts (1994) or Topkis (1998, p.41, Theorem 2.5.2).

Proof. See Appendix. ■

Secondly, in a pseudo-potential game, for each pure strategy Nash equilibrium, we can find a pseudo-potential such that a unique maximizer of the pseudo-potential is the Nash equilibrium:

Lemma 12 *Let $\mathbf{g}^N$ be a pseudo-potential game. If a^* is a pure strategy Nash equilibrium of $\mathbf{g}^N$, then there exists a pseudo-potential $f : A \to \mathbb{R}$ such that $\{a^*\} = \arg\max_{a \in A} f(a)$.*

Proof. See Appendix. ■

Thirdly, we have the following characterization of partition pseudo-potentials by the definition of the parition pseudo-potentials:

Lemma 13 *$(f_T)_{T \in \mathcal{T}}$ is a partition $\mathcal{T}$ pseudo-potential of $\mathbf{g}^N$ if and only if, for each member T of $\mathcal{T}$, for any action $a_{-T} \in A_{-T}$ of all players outside T, $f_T(\cdot, a_{-T})$ is a pseudo-potential of the restricted game $\mathbf{g}^N|_{a_{-T}}$ by a_{-T}.*

Finally, a finite two-person game of weak strategic complementarities has a pseudo-potential. Indeed, Dubey *et al.* (2006) showed that a two-person finite game of weak strategic complementarities, where each action set is one-dimensional, has a pseudo-potential:[10]

[10] In fact, Dubey *et al.* (2006) showed that games with aggregators of weak strategic complementarities or weak substitutes are pseudo-potential games.

Proposition 14 (Dubey *et al*, 2006) *Let* $\mathbf{g}^{\{1,2\}}$ *be a two-person finite game with* $A_1, A_2 \subset \mathbb{R}$*. If* $\mathbf{g}^{\{1,2\}}$ *has weak strategic complementarities, then it is a pseudo-potential game.*

We extend Proposition 14 to the case where the action set of one player is multi-dimensional.

Proposition 15 *Let* $\mathbf{g}^{\{1,2\}}$ *be a finite two-person game with* $A_1 \subset \mathbb{R}^m$*, where* $m \in \mathbb{N}$*, and* $A_2 \subset \mathbb{R}$*. If* $\mathbf{g}^{\{1,2\}}$ *has weak strategic complementarities, then it is a pseudo-potential game.*

Proof. See Appendix. ■

We prove Theorem 10 by applying Lemmas 11, 12, 13 and Proposition 15 iteratively. The outline of the proof is followings: let $\mathbf{g}^N$ be a game of weak strategic complementarities. Firstly, by Lemma 13 and Proposition 15, we know there exists a partition $\{\{1,2\},3,\ldots,n\}$ pseudo-potential of $\mathbf{g}^N$. Next, by Lemmas 11 and 12, in particular, we can find a partition $\{\{1,2\},3,\ldots,n\}$ pseudo-potential $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ such that a best-response of representative agent $\{1,2\}$ is increasing with $a_{-\{1,2\}}$. Then, we can regard $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ as a games of weak strategic complementarities. Moreover, by applying Lemmas 11, 12, 13 and Proposition 15, we have a partition $\{\{1,2,3\},\ldots,n\}$ pseudo-potential $\mathbf{f}^{\{\{1,2,3\},\ldots,n\}}$ of $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ such that $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ is a game of weak strategic complementarities, and so on. Finaly, we can find a partition $\{\{1,2,\ldots,n\}\}$ pseudo-potential ($f^{\{\{1,2,\ldots,n\}\}}$) of $\mathbf{f}^{\{\{1,2,\ldots,n-1\},n\}}$. and thus, we have a nested pseudo-potential $f = f^{\{\{1,2,\ldots,n\}\}}$.

Proof of Theorem 10. See Appendix. ■

6 Examples

When the action set of a single player is allowed multi-dimensional, in what follows, we show by way of examples that the relationship among strategic complementarities, a pseudo potential and a nested pseudo potential is given as in Figure 1.

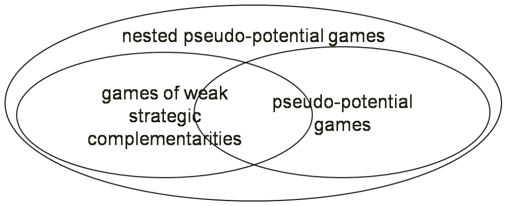

Figure 1: Strategic complementarities and nested potential games when the action set of one player is multi-dimensional and the action sets of the others are one-dimensional

As mentioned in Proposition 14, Dubey *et al.* (2006) show that two person games of weak strategic complementarities is a pseudo-potential game. However, games with more than two players of weak strategic complementarities may not be a pseudo-potential game as in the following example.

Example 16 Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 1, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

0	0	1	1	0	1
0	$4,4,4$	$0,0,1$	0	$1,0,0$	$0,1,0$
1	$0,1,0$	$1,0,0$	1	$0,0,1$	$4,4,4$

Table 1: (g_1, g_2, g_3)

We can show that $\mathbf{g}^{\{1,2,3\}}$ has weak strategic complementarities.

However, this game is not a pseudo-potential game. Indeed, note that $\mathbf{g}^{\{1,2,3\}}$ has a strict best-response cycle $(1,0,0) \to (1,0,1) \to (0,0,1) \to (0,1,1) \to (0,1,0) \to (1,1,0) \to (1,0,0)$. Since pseudo-potential games cannot have strict best-response cycles as shown by Schipper (2004), this game is not a pseudo-potential game. On the other hand, $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game. Indeed, $(f^2_{\{1,2\}}, f^1_{\{3\}})$ given in Table 2 is a $\{\{1,2\},\{3\}\}$-pseudo-potential of $\mathbf{g}^{\{1,2,3\}}$, where $f^1_{\{3\}}(\cdot) = g_3(\cdot)$.

Regarding the $\{\{1,2\},\{3\}\}$-pseudo-potential $(f^2_{\{1,2\}}, f^1_{\{3\}})$ as a strategic form game, we can show that $(f^2_{\{1,2,3\}})$ defined in Table 3 is a $\{\{1,2,3\}\}$-pseudo-potential of $(f^2_{\{1,2\}}, f^1_{\{3\}})$. Thus

	0	1
(0, 0)	3, 4	1, 0
(0, 1)	0, 1	2, 0
(1, 0)	2, 0	0, 1
(1, 1)	1, 0	3, 4

Table 2: $(f^1_{\{1,2\}}, f^1_{\{3\}})$

	a_3	1
(0, 0)	2	0
(0, 1)	1	0
(1, 0)	0	1
(1, 1)	0	2

Table 3: $f^2_{\{1,2,3\}}$ or f

$\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game.

A pseudo-potential game may not have strategic complementarities as in the following example.

Example 17 Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 5, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

0	0	1	2	1	0	1	2
0	0, 0, 0	1, 1, 0	0, 0, 1	0	1, 1, 1	1, 0, 1	0, 0, 0
1	0, 0, 0	0, 0, 1	1, 1, 0	1	1, 0, 1	0, 0, 0	1, 0, 1

Table 4: $\mathbf{g}^{\{1,2,3\}}$ is a pseudo-potential without weak strategic complementarities

We can show that $\mathbf{g}^{\{1,2,3\}}$ has a pseudo-potential f in Table 5. We can also show that $\mathbf{g}^{\{1,2,3\}}$ does not have weak strategic complementarities.

0	0	1	2	1	0	1	2
0	0	2	1	0	4	3	0
1	0	1	2	1	1	0	3

Table 5: a pseudo-potential f of $\mathbf{g}^{\{1,2,3\}}$

The following game, which appeared in Uno (2007a), strategic complementarities or a pseudo-potential game but it is a nested pseudo-potential game.

Example 18 (Uno, 2007a) Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ in Table 6, where player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix;

players 1 and 2 have identical interests, player 3's payoff is the same as others when player 1 chooses a_1, but is reversed otherwise as in the matching pennies game.

0	0	1	1	0	1
0	$3,3,3$	$0,0,0$	0	$0,0,0$	$2,2,2$
1	$-1,-1,1$	$1,1,-1$	1	$1,1,-1$	$-1,-1,1$

Table 6: (g_1, g_2, g_3)

Note that $\mathbf{g}^{\{1,2,3\}}$ is not a game of strategic complementarities. Note also that $\mathbf{g}^{\{1,2,3\}}$ is not a pseudo-potential game. Indeed, $\mathbf{g}^{\{1,2,3\}}$ has a strict best-response cycle $(1,0,0) \to (1,1,0) \to (1,1,1) \to (1,0,1) \to (1,0,0)$. Since pseudo-potential games cannot have strict best response cycles as shown by Schipper (2004), the game is not a pseudo-potential game. However, we can show that $\mathbf{g}^{\{1,2,3\}}$ is a nested pseudo-potential game.

The next example demonstrates the failure of Proposition 14 when the each action set of more than one player is multi-dimensional.

Example 19 Consider the two-person game $\mathbf{g}^{\{1,2\}}$ represented as Table 7, where $A_1 = A_2 = \{0,1\} \times \{0,1\}$, player 1 chooses the row and the column, and player 2 chooses the matrix. We can show that game (g_1, g_2) has weak strategic complementarities. But it

$(0,0)$	0	1	$(0,1)$	0	1	$(1,0)$	0	1	$(1,1)$	0	1
0	$1,1$	$0,0$	0	$0,0$	$0,1$	0	$0,0$	$1,0$	0	$0,0$	$0,0$
1	$0,0$	$0,0$	1	$1,0$	$0,0$	1	$0,1$	$0,0$	1	$0,0$	$1,1$

Table 7: (g_1, g_2)

does not have a nested pseudo-potential. Indeed, there exists a strict best-response cycle $((0,1),(0,1)) \to ((1,0),(0,1)) \to ((1,0),(1,0)) \to ((0,1),(1,0)) \to ((0,1),(0,1))$. So, by Schipper (2004), this game is not a pseudo-potential game.

Remark 20 Dubey *et al.* (2006) presented a more general version of Proposition 14 where action sets are compact subsets of $\mathbb{R}$, provided that, for each player i, i's best-response selection b_i is continous on the set A_{-i} of the other players' action profiles additionally.

But, we cannot immediately extend Theorem 10 to games with compact action sets. This is because it is difficult to guarantee that there exists a partition potential $\mathbf{f}^{\{\{1,2\},3,\ldots,n\}}$ of a game $\mathbf{g}^N$ such that a best-response selection $b_{\{1,2\}}$ of representative agent $\{1,2\}$ is continous on the set $A_{-\{1,2\}}$ of action profiles of players outside $\{1,2\}$, since game of weak strategic complementarities does not always have a continuous increasing equilibrium selection.

7 A Generalization of Strategic Complementarities and Nested Potential Games

If the action sets of two or more players are multi-dimensional, there is no relationship between the games of weak strategic complementarities and the nested potential games, as shown in Example 18 and Example 19. This section proposes a new class of games which possess a pure strategy Nash equilibrium. To do so, we introduce the nested partition pseudo-potential. A nested partition pseudo-potential is a partition pseudo-potential when we give up the construction of the nested pseudo-potential halfway:

Definition 21 Let $\mathcal{T}$ be a partition of N. A tuple $\mathbf{f}^{\mathcal{T}} = (\mathcal{T}, (A_T)_{T\in\mathcal{T}}, (f_T)_{T\in\mathcal{T}})$ is a *nested partition* $\mathcal{T}$ *pseudo-potential* of $\mathbf{g}^N$, if there exist a positive integer K and a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=1}^K = ((f_T^k)_{T\in\mathcal{T}^k})_{k=1}^K$ such that

- $(\mathcal{T}^k)_{k=1}^K$ is an increasingly coarser sequence of partitions of N with $\mathcal{T}^1 = \{\{i\}|i \in N\}$ and $\mathcal{T}^K = \mathcal{T}$;
- $\mathbf{f}^{\mathcal{T}^1}$ is regarded as the original game $\mathbf{g}^N$: for each $i \in N$, $f_{\{i\}}^1(a) = g_i(a)$ for all $a \in A$;
- $\mathbf{f}^{\mathcal{T}^K} = \mathbf{f}^{\mathcal{T}}$: for any $T \in \mathcal{T}$, $f_T^K(a) = f_T(a)$ for all $a \in A$.
- for each $k = 2, 3, \ldots, K$, $\mathbf{f}^{\mathcal{T}^k} = (f_T^k)_{T\in\mathcal{T}^k}$ is a $\mathcal{T}^k$-pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}} = (f_T^{k-1})_{T\in\mathcal{T}^{k-1}}$, where $\mathbf{f}^{\mathcal{T}^{k-1}}$ is regarded as a strategic form game as above: for each

$T^k \in \mathcal{T}^k$ and each $T^{k-1} \in \mathcal{T}^{k-1}$ with $T^{k-1} \subseteq T^k$,

$$\arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} f^k_{T^k}(a_{T^{k-1}}, a_{-T^{k-1}}) \subseteq \arg \max_{a_{T^{k-1}} \in A_{T^{k-1}}} f^{k-1}_{T^{k-1}}(a_{T^{k-1}}, a_{-T^{k-1}})$$

for all $a_{-T^{k-1}} \in A_{-T^{k-1}}$.

If a nested partition potential of a strategic form game $\mathbf{g}^N$ has a pure strategy Nash equilibrium a^*, then the game $\mathbf{g}^N$ also have a pure strategy Nash equilibrium a^*:

Theorem 22 *Let $\mathcal{T}$ be a partition of N. Let $\mathbf{f}^{\mathcal{T}}$ be a nested partition $\mathcal{T}$ pseudo-potential of a strategic form game $\mathbf{g}^N$. If an action profile $a^* \in A$ is a pure strategy Nash equilibrium of $\mathbf{f}^{\mathcal{T}}$, then a^* is also a pure strategy Nash equilibrium of $\mathbf{g}^N$.*

We can prove Theorem 22 by applying Lemma 2.9 of Uno (2007a) iteratively.

We provide a necessary condition for the existence of a nested partition $\mathcal{T}$ pseudo-potentials. We use this condition to demonstrate that a game does not have a nested pseudo-potential below.

If a game has a nested partition $\mathcal{T}$ pseudo-potential and the number of elements of $\mathcal{T}$ is L, then we can find a sequence of partitions which reaches $\mathcal{T}$ from $\{i|i \in N\}$ in $n-L$ steps in the following way: at each step k, only two elements T^k_1 and T^k_2 of the partition $\mathcal{T}^k$ are united; and, for any action profile $a_{T^k_2}$ of members of T^k_2, there exists a pure strategy Nash equilibrium selection $e_{T^k_1}(a_{T^k_2})$ of $\mathbf{f}|_{a_{-T^k_2}}$ and, for any action profile $a_{T^k_1}$ of T^k_1's members, there exists a pure strategy Nash equilibrium selection $e_{T^k_2}(a_{T^k_1})$ of $\mathbf{f}|_{a_{-T^k_1}}$ such that a cycle by $e_{T^k_1}$ and $e_{T^k_2}$ does not happen:

Lemma 23 *If a game $\mathbf{g}^N$ has a nested partition $\mathcal{T}$ pseudo-potential and the number of elements of $\mathcal{T}$ is L, then there exists a sequence of partitions $(\mathcal{T}^k)^{n-L}_{k=1}$ such that*

- *$(\mathcal{T}^k)^{n-L}_{k=1}$ is an unit increasingly coarser sequence of partitions of N: $\mathcal{T}^1 = \{\{i\}|i \in N\}$, $\mathcal{T}^{n-L} = \mathcal{T}$, and, for each $k = 2, 3, \ldots, n-L$, there exist $T^k_1, T^k_2 \in \mathcal{T}^k$ such that $\mathcal{T}^{k+1} \backslash \mathcal{T}^k = \{T^k_1 \cup T^k_2\} \backslash \{T^k_1, T^k_2\}$; and*

- *for each $k = 2, 3, \ldots, n-L$ and for each $j = 1, 2$, there exists a function $e_{T_j^k} : A_{-T_j^k} \to A_{T_j^k}$ such that*

  - *$e_{T_j^k}$ is an equilibrium selection: for any $a_{-T_j^k} \in A_{-T_j^k}$, $e_{T_j^k}(a_{-T_j^k})$ is a pure strategy Nash equilibrium of $\mathbf{g}^N|_{a_{-T_j^k}}$, and*
  - *for any $a_{-T_1^k \cup T_2^k} \in A_{-T_1^k \cup T_2^k}$ and for any $a_l^k \in A_l^k$, where $l \in \{1, 2\}$ with $l \neq j$,*

$$e_{T_j^k}(e_{T_l^k}(e_{T_j^k}(a_{T_l^k}, a_{-T_1^k \cup T_2^k}), a_{-T_1^k \cup T_2^k}), a_{-T_1^k \cup T_2^k}) \neq e_{T_j^k}(a_{T_l^k}, a_{-T_1^k \cup T_2^k}).$$

Proof. See Appendix. ■

We introduce the special class of games with nested partition potential:

Definition 24 A game has *nested weak strategic complementarities* if there exist a partition $\mathcal{T}$ of N and a nested partition $\mathcal{T}$ pseudo-potential $\mathbf{f}^{\mathcal{T}}$ such that $\mathbf{f}^{\mathcal{T}}$ has weak strategic complementarities.

Note that games of the nested weak strategic complementarities generalize the games of weak strategic complementarities and the nested potential games: indeed, it is clear that every game of weak strategic complementarities is a game of nested weak strategic complementarities; it is also clear that every nested potential game is a game of nested weak strategic complementarities since we can regard one person game as game of weak strategic complementarities, and any Nash equilibrium of $\mathbf{f}^{\mathcal{T}}$ is also a Nash equilibrium of the original game by Theorem 23.

Corollary 25 below states that implies games of nested weak strategic complementarities have a pure strategy Nash equilibrium. This result holds, since there exists a pure strategy Nash equilibrium in games of weak strategic complementarities:

Corollary 25 *If $\mathbf{g}^N$ has nested weak strategic complementarities, then there exists a pure strategy Nash equilibrium of $\mathbf{g}^N$.*

Even if a game has neither weak strategic complementarities nor a nested pseudo-potential, it may have nested weak strategic complementarities, as shown in the following example.

Example 26 Consider the three-person game $\mathbf{g}^{\{1,2,3\}}$ represented as Table 8, where $A_1 = A_2 = \{0,1,2\}$, and $A_3 = \{0,1\} \times \{0,1\}$, player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix.

$(0,0)$	0	1	2
0	$1,1,1$	$1,0,0$	$1,0,0$
1	$0,1,0$	$0,0,0$	$0,0,0$
2	$0,1,0$	$0,1,0$	$0,0,0$

$(0,1)$	0	1	2
0	$1,0,0$	$1,0,0$	$1,1,0$
1	$0,0,1$	$0,1,0$	$0,0,0$
2	$0,0,1$	$0,0,0$	$0,1,0$

$(1,0)$	0	1	2
0	$0,1,0$	$0,0,1$	$0,0,1$
1	$0,1,0$	$0,0,0$	$0,0,0$
2	$1,1,0$	$1,0,0$	$1,0,0$

$(1,1)$	0	1	2
0	$1,1,0$	$1,0,0$	$0,0,0$
1	$0,1,0$	$0,0,1$	$0,0,1$
2	$0,0,0$	$0,0,1$	$1,1,1$

Table 8: $\mathbf{g}^{\{1,2,3\}}$

We can show that game $\mathbf{g}^{\{1,2,3\}}$ dose not have weak strategic complementarities. Moreover, $\mathbf{g}^{\{1,2,3\}}$ has neither weak strategic complementarities nor a nested pseudo-potential. Indeed, if $a_1 = 1$, there exists a strict best-response cycle $(1,0,(0,1)) \to (1,1,(0,1)) \to (1,1,(1,1)) \to (1,0,(1,1)) \to (1,0,(0,1))$; and, if $a_2 = 1$, there exists a strict best-response cycle $(0,1,(1,0)) \to (2,1,(1,0)) \to (2,1,(1,1)) \to (0,1,(1,1)) \to (0,1,(1,0))$. So, (g_1, g_2, g_3) has $\{\{1,2\},\{3\}\}$-pseudo-potential, but neither $\{\{1\},\{2,3\}\}$-pseudo-potential nor $\{\{2\},\{1,3\}\}$-pseudo-potential by Lemma 23.

And, for any a_3, there exists no strict best-response cycle. But if $a_3 = (1,0)$, $\mathbf{g}^{\{1,2,3\}}|_{a_3}$ has a unique pure strategy Nash equilibrium $(a_1,a_2) = (2,0)$; and if $a_3 = (0,1)$, $\mathbf{g}^{\{1,2,3\}}|_{a_3}$ has also a unique pure strategy Nash equilibrium $(a_1,a_2) = (0,2)$. So, $\{\{1,2\},\{3\}\}$-pseudo-potential must have a strict best-response cycle $(2,0,(1,0)) \to (2,0,(0,1)) \to (0,2,(0,1)) \to (0,2,(1,0)) \to (2,0,(1,0))$. By Lemma 23, $\{\{1,2\},\{3\}\}$-pseudo-potential have not a pseudo-potential. That is, $\mathbf{g}^{\{1,2,3\}}$ is not a nested pseudo-potential game.

However, $\mathbf{g}^{\{1,2,3\}}$ has nested weak strategic complementarities. Indeed, $(f^2_{\{1,2\}}, f^1_{\{3\}})$ given in Table 9 is a $\{\{1,2\},\{3\}\}$-pseudo-potential of $\mathbf{g}^{\{1,2,3\}}$, where $f^1_{\{3\}}(\cdot) = g_3(\cdot)$. Regarding the $\{\{1,2\},\{3\}\}$-pseudo-potential $(f^1_{\{1,2\}}, f^1_{\{3\}})$ as a strategic form game, we can show that $(f^1_{\{1,2\}}, f^1_{\{3\}})$ has weak strategic complementarities. Thus, $\mathbf{g}^{\{1,2,3\}}$ has nested weak strategic complementarities.

$(0,0)$	0	1	2
0	$2,1$	$1,0$	$1,0$
1	$1,0$	$0,0$	$0,0$
2	$1,0$	$0,0$	$0,0$

$(0,1)$	0	1	2
0	$1,0$	$2,0$	$3,0$
1	$0,1$	$1,0$	$0,0$
2	$0,1$	$0,0$	$1,0$

$(1,0)$	0	1	2
0	$1,0$	$0,1$	$0,1$
1	$1,0$	$0,0$	$0,0$
2	$2,0$	$1,0$	$1,0$

$(1,1)$	0	1	2
0	$2,0$	$1,0$	$0,0$
1	$1,0$	$0,1$	$0,1$
2	$0,0$	$0,1$	$1,1$

Table 9: $(f^1_{\{1,2\}}, f^1_{\{3\}})$

By Examples 18, 19 and 26, If the action sets of two or more players are multi-dimensional, the relationship among Nested strategic complementarities, strategic complementarities and nested potential games is as depicted in Figure 2.

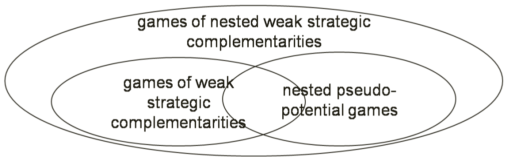

Figure 2: Nested strategic complementarities, strategic complementarities and nested potential games when the action sets of two or more players are multi-dimensional

8 Concluding Remarks

This paper investigates the relationship between the games of weak strategic complementarities and the nested potential games. For this, we introduce a new class of games generalizing games of weak strategic complementarities and nested potential games.

This result suggests the following. If the action set of one player is multi-dimensional and the action sets of the others are one-dimensional, it is sufficient to check whether the game has a nested pseudo-potential. If the action sets of two or more players are multi-dimensional, we should check whether the game has a nested partition pseudo-potential or weak strategic complementarities iteratively.

Appendix: Proofs

Proof of Lemma 11. Suppose that $\mathbf{g}^N$ is a game of weak strategic complementarities. For each $i \in N$, let $b_i : A_{-i} \to A_i$ be i's best-response selection such that $b_i(a_{-i}) \leq b_i(a'_{-i})$ whenever $a_{-i} < a'_{-i}$. Fix any $T \subseteq N$. For any $a_{-T} \in A_{-T}$, let $b_T(\cdot, a_{-T}) : A_T \to A_T$ be the function defined by $b_T(a_T, a_{-T}) := (b_i(a_{T\setminus\{i\}}, a_{-T}))_{i\in T}$ for any $a_T \in A_T$. For any $a_{-T} \in A_{-T}$, since $b_T(\cdot, a_{-T})$ is an increasing function, by Tarski's fixed point theorem (Theorem 1), there exists the greatest (least) fixed point of $b_T(\cdot, a_{-T})$, i.e., the greatest (least) pure strategy Nash equilibrium of $\mathbf{g}^N|_{a_{-T}}$.

Pick any $a_{-T}, a'_{-T} \in A_{-T}$ with $a_{-T} < a'_{-T}$. Let $e_T(a_{-T})$ and $e_T(a'_{-T})$ be the greatest pure strategy Nash equilibria of $\mathbf{g}^N|_{a_{-T}}$ and $\mathbf{g}^N|_{a'_{-T}}$, respectively. Because $e_T(a_{-T}) = b_T(e_T(a_{-T}), a_{-T})$ and $b_T(e_T(a_{-T}), a_{-T}) \leq b_T(e_T(a_{-T}), a'_{-T})$, we have $e_T(a_{-T}) \leq b_T(e_T(a_{-T}), a'_{-T})$. By Theorem 1, $\sup\{a_T \in A_T | a_T \leq b_T(a_T, a'_{-T})\}$ is the greatest pure strategy Nash equilibrium of $\mathbf{g}^N|_{a'_{-T}}$. Thus, we have $e_T(a_{-T}) \leq e_T(a'_{-T})$. ■

Proof of Proposition 15. Suppose that $\mathbf{g}^{\{1,2\}}$ has weak strategic complementarities. Then, for $i, j = 1, 2$ with $i \neq j$, there exists a function $b_i : A_j \to A_i$ such that $b_i(a_j) \in \arg\max_{a_i\in A_i} g_i(a_i, a_j)$ for all $a_j \in A_j$, and $b_i(a_j) \leq b_i(a'_j)$ whenever $a_j < a'_j$. Let A'_1 be the range of b_1, i.e., $A'_1 := \{a_1 \in A_1 |$ there exists $a_2 \in A_2$ such that $a_1 = b_1(a_2)\}$. Since A'_1 is linearly ordered and finite, there exist a subset A_1 of $\mathbb{R}$ and a bijection h from A'_1 to A_1 such that for each $a_1, a'_1 \in A'_1$, $a_1 < a'_1$ if and only if $h(a_1) < h(a'_1)$. Such $\hat{A}_1$ exists by the property of b_1. Let $\hat{g}_1 : \hat{A}_1 \times A_2 \to \mathbb{R}$ be the function defined by $\hat{g}_1(\hat{a}_1, a_2) = g_1(h^{-1}(\hat{a}_1), a_2)$ for all $\hat{a}_1 \in \hat{A}_1$ and all $a_2 \in A_2$.

Consider a two-person game $(\hat{g}_1, g_2)$ given by $(\hat{g}_1, g_2) := (\{1, 2\}, (\hat{A}_1, A_2), (\hat{g}_1, g_2))$. It then follows that there exists player 1's best-response selection $\hat{b}_1 : A_2 \to \hat{A}_1$ such that $\hat{b}_1(a_2) \leq \hat{b}_1(a'_2)$ whenever $a_2 < a'_2$, since A'_1 and $\hat{A}_1$ are order isomorphic, $b_1(a_2) \in A'_1$ for any $a_2 \in A_2$, and $b_1(a_2) \leq b_1(a'_2)$ whenever $a_2 < a'_2$. Since $\mathbf{g}^{\{1,2\}}$ has strategic complementarities, there exists also player 2's best-response selection $\hat{b}_2 : \hat{A}_1 \to A_2$ such that $\hat{b}_2(a_1) \leq \hat{b}_2(a'_1)$

whenever $a_1 < a_1'$. Thus, $(\hat{g}_1, g_2)$ has weak strategic complementarities. By proposition 14, $(\hat{g}_1, g_2)$ has a pseudo-potential $\hat{f} : \hat{A}_1 \times A_2 \to \mathbb{R}$.

Let $c \in \mathbb{R}$ be sufficiently small so that $c < \min_{(\hat{a}_1, a_2) \in \hat{A}_1 \times A_2} \hat{f}(\hat{a}_1, a_2)$, which exists since $\hat{A}_1 \times A_2$ is finite.

Let $f : A \to \mathbb{R}$ be a function such that, for all $a_1 \in A_1$ and all $a_2 \in A_2$,

$$f(a_1, a_2) = \begin{cases} \hat{f}(h(a_1), a_2) & \text{if } a_1 \in A_1' \\ c & \text{if } a_1 \in A_1 \backslash A_1' \text{ and } a_2 \in b_2(a_1) \\ c - 1 & \text{otherwise} \end{cases} \tag{2}$$

We will show that f is a pseudo-potential of $\mathbf{g}^{\{1,2\}}$. Fix any $a_2 \in A_2$. Pick any $a_1^{**} \in \arg\max_{a_1 \in A_1} f(a_1, a_2)$. Then, $a_1^{**} \in A_1'$ must hold by the choice of constant c in the construction of f. Since $a_1^{**} \in \arg\max_{a_1 \in A_1'} f(a_1, a_2)$, we have $h(a_1^{**}) \in \arg\max_{\hat{a}_1 \in \hat{A}_1} \hat{f}(\hat{a}_1, a_2)$. Since $\hat{f}$ is a pseudo-potential of $(\hat{g}_1, g_2)$, we have $h(a_1^{**}) \in \arg\max_{\hat{a}_1 \in \hat{A}_1} \hat{g}_1(\hat{a}_1, a_2)$. Since A_1' and $\hat{A}_1$ are order isomorphic, we have $a_1^{**} \in \arg\max_{a_1 \in A_1'} g_1(a_1, a_2)$. And, since $a_1^{**} \in A_1'$, we have $g_1(a_1^{**}, a_2) \geq g_1(a_1, a_2)$ for all $a_1 \in A_1 \backslash A_1'$. Thus, we have $a_1^{**} \in \arg\max_{a_1 \in A_1} g_1(a_1, a_2)$.

Fix any $a_1 \in A_1$. Pick any $a_2^{**} \in \arg\max_{a_2 \in A_2} f(a_1, a_2)$. If $a_1 \in A_1'$, we have $a_2^{**} \in \arg\max_{a_2 \in A_2} g_2(a_1, a_2)$, since $\hat{f}$ is a pseudo-potential of $(\hat{g}_1, g_2)$. If $a_1 \in A_1 \backslash A_1'$, we must have $a_2^{**} \in b_2(a_1)$ by the construction of f. Thus, we have $a_2^{**} \in \arg\max_{a_2 \in A_2} g_2(a_1, a_2)$. Hence, f is a pseudo-potential of $\mathbf{g}^{\{1,2\}}$. ■

Proof of Lemma 12. Suppose that $\mathbf{g}^N$ is a game with pseudo-potential f. Let a^* be a pure strategy Nash equilibrium of $\mathbf{g}^N$. Let $c \in \mathbb{R}$ be a sufficiently large number such that $c > \max_{a \in A} f(a)$. Define a function $\hat{f} : A \to \mathbb{R}$ such that, for each $a \in A$,

$$\hat{f}(a) = \begin{cases} c & \text{if } a = a^* \\ f(a) & \text{otherwise} \end{cases}$$

Then, we have $\{a^*\} = \arg\max_{a \in A} \hat{f}(a)$. And, we can show that $\hat{f}$ is also a pseudo-

potential of $\mathbf{g}^N$. Indeed, fix any $i \in N$ and any $a_{-i} \in A_{-i}$. If $a_{-i} \neq a^*_{-i}$, we have $\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i}) = \arg\max_{a_i \in A_i} f(a_i, a_{-i})$. Since f is a pseudo-potential of $\mathbf{g}^N$, $\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. If $a_{-i} = a^*_{-i}$, we have $\{a^*_i\} = \arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i})$. Since a^* is a Nash equilibrium, we have $a^*_i \in \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. Thus we have $\arg\max_{a_i \in A_i} \hat{f}(a_i, a_{-i}) \subseteq \arg\max_{a_i \in A_i} g_i(a_i, a_{-i})$. Hence $\hat{f}$ is a pseudo-potential of $\mathbf{g}^N$.
■

Proof of Theorem 10. Without loss of generality, we will assume that $m_1 \in \mathbb{N}$ and $m_i = 1$ for each $i \neq 1$. Suppose that $\mathbf{g}^N$ is a game of weak strategic complementarities. We shall show that, for each $l = 1, 2, \ldots, n$, there exists a function $f^l_{\{1,\ldots,l\}} : A \to \mathbb{R}$ such that

1. $(f^l_{\{1,\ldots,l\}}, g_{l+1}, \ldots, g_n)$ is a $\{\{1,\ldots,l\},\{l+1\},\ldots,\{n\}\}$-pseudo-potential of $(f^{l-1}_{\{1,\ldots,l-1\}}, g_l, \ldots, g_n)$, where $(f^0_{\{0\}}, g_1, \ldots, g_n) := (g_1, \ldots, g_n)$;

2. there exists a function $b_{\{1,\ldots,l\}} : A_{-\{1,\ldots,l\}} \to A_{\{1,\ldots,l\}}$ with

   - $b_{\{1,\ldots,l\}}(a_{-\{1,\ldots,l\}}) \in \arg\max\limits_{a_{\{1,\ldots,l\}} \in A_{\{1,\ldots,l\}}} f^l_{\{1,\ldots,l\}}(a_{\{1,\ldots,l\}}, a_{-\{1,\ldots,l\}})$ for all $a_{-\{1,\ldots,l\}} \in A_{-\{1,\ldots,l\}}$, and
   - $b_{\{1,\ldots,l\}}(a_{-\{1,\ldots,l\}}) \geq b_{\{1,\ldots,l\}}(a'_{-\{1,\ldots,l\}})$ whenever $a_{-\{1,\ldots,l\}} > a'_{-\{1,\ldots,l\}}$.

The proof proceeds by induction on l. First, when $l = 1$, let $f^1_{\{1\}}(\cdot) := g_1(\cdot)$. Then, $(f^1_{\{1\}}, \ldots, g_n)$ is a $\{\{1\},\ldots,\{n\}\}$-potential of $(g_1, \ldots, g_n)$. Moreover, since $\mathbf{g}^N$ is a game of weak strategic complementarities, there exists a function $b_{\{1\}} : A_{-\{1\}} \to A_{\{1\}}$ with $b_{\{1\}}(a_{-\{1\}}) \in \arg\max\limits_{a_{\{1\}} \in A_{\{1\}}} f^1_{\{1\}}(a_{\{1\}}, a_{-\{1\}})$ for all $a_{-\{1\}} \in A_{-\{1\}}$, and $b_{\{1\}}(a_{-\{1\}}) \geq b_{\{1\}}(a'_{-\{1\}})$ whenever $a_{-\{1\}} > a'_{-\{1\}}$.

Suppose that, for each $l \leq k-1 \leq n-1$, there exist functions $f^l_{\{1,\ldots,l\}} : A \to \mathbb{R}$ and $b_{\{1,\ldots,l\}} : A_{-\{1,\ldots,l\}} \to A_{\{1,\ldots,l\}}$ such that the conditions 1 and 2 hold. We will show that there exists such functions $f^k_{\{1,\ldots,k\}} : A \to \mathbb{R}$ and $b_{\{1,\ldots,k\}} : A_{-\{1,\ldots,k\}} \to A_{\{1,\ldots,k\}}$.

Fix any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$. Consider a restricted game $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ by $a_{-\{1,\ldots,k\}}$. By the assumption of induction, there exists a function $b_{\{1,\ldots,k-1\}}(\cdot, a_{-\{1,\ldots,k\}}) : A_k \to A_{\{1,\ldots,k-1\}}$ with $b_{\{1,\ldots,k-1\}}(a_k, a_{-\{1,\ldots,k\}}) \in \arg\max\limits_{a_{\{1,\ldots,k-1\}} \in A_{\{1,\ldots,k-1\}}} f^{k-1}_{\{1,\ldots,k-1\}}(a_{\{1,\ldots,k-1\}}, a_k, a_{-\{1,\ldots,k\}})$ for all $a_k \in A_k$, and $b_{\{1,\ldots,k-1\}}(a_k, a_{-\{1,\ldots,k\}}) \geq b_{\{1,\ldots,k-1\}}(a'_k, a_{-\{1,\ldots,k\}})$ whenever $a_k > a'_k$.

And, since b_k is player k's best-response selection such that $b_k(a_{\{1,\ldots,k-1\}}, a_{-\{1,\ldots,k\}}) \geq b_k(a'_{\{1,\ldots,k-1\}}, a_{-\{1,\ldots,k\}})$ whenever $a_{\{1,\ldots,k-1\}} > a'_{1,\ldots,k-1}$, we can regard the restricted game $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ by $a_{-\{1,\ldots,k\}}$ as a two-person game of weak strategic complementarities, where $N = \{\{1,\ldots,k-1\},\{k\}\}$, $A_{\{1,\ldots,k\}} \subset \mathbb{R}^{m+k-1}$, and $A_k \subset \mathbb{R}$. By Proposition 15, $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, g_{k+1}, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ has a pseudo-potential.

Now, consider a restricted game $\mathbf{g}^N|_{a_{-\{1,\ldots,k\}}}$ for any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$. Since $\mathbf{g}^N$ has weak strategic complementarities, by Lemma 11, there exists an equilibrium selection $e_{\{1,\ldots,k\}} : A_{-\{1,\ldots,k\}} \to A_{\{1,\ldots,k\}}$ of $\mathbf{g}^N|_{a_{-\{1,\ldots,k\}}}$ such that $e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}}) \leq e_{\{1,\ldots,k\}}(a'_{-\{1,\ldots,k\}})$ whenever $a_{-\{1,\ldots,k\}} < a'_{-\{1,\ldots,k\}}$.

For any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$, since $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, \ldots, g_n)|_{a_{-\{1,\ldots,k\}}}$ has a pseudo-potential, by Lemma 12, there exists a pseudo-potential $f^k_{\{1,\ldots,k\}}(\cdot, a_{-\{1,\ldots,k\}}) : A_{\{1,\ldots,k\}} \to \mathbb{R}$ such that $e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}})$ is a unique maximizer of $f^k_{\{1,\ldots,k\}}$:

$$\{e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}})\} = \arg \max_{a_{\{1,\ldots,k\}} \in A_{\{1,\ldots,k\}}} f^k_{\{1,\ldots,k\}}(a_{\{1,\ldots,k\}}, a_{-\{1,\ldots,k\}}). \tag{3}$$

Recall that, for any partition $\mathcal{T}$ of N, $\mathbf{g}^N$ has a partition $\mathcal{T}$ pseudo-potential if and only if, for each member T of $\mathcal{T}$, for any $a_{-T} \in A_{-T}$, the restricted game $\mathbf{g}^N|_{a_{-T}}$ by a_{-T} is a pseudo-potential game (Definition 3). For any partition $\mathcal{T}$ of N, recall that $(f_T)_{T\in\mathcal{T}}$ is a partition $\mathcal{T}$ pseudo-potential of $\mathbf{g}^N$ if and only if, for each member T of $\mathcal{T}$, for any $a_{-T} \in A_{-T}$, $f_T(\cdot, a_{-T})$ is a pseudo-potential of the restricted game $\mathbf{g}^N|_{a_{-T}}$ by a_{-T}. Thus, $(f^k_{\{1,\ldots,k\}}, g_{k+1}, \ldots, g_n)$ is a $\{\{1,\ldots,k\},\{k+1\},\ldots,\{n\}\}$-pseudo-potential of $(f^{k-1}_{\{1,\ldots,k-1\}}, g_k, g_{k+1}, \ldots, g_n)$. Hence, $f^k_{\{1,\ldots,k\}}$ satisfies Condition 1.

Let $b_{\{1,\ldots,k\}} : A_{-\{1,\ldots,k\}} \to A_{\{1,\ldots,k\}}$ be the function defined by $b_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}}) := e_{\{1,\ldots,k\}}(a_{-\{1,\ldots,k\}})$ for any $a_{-\{1,\ldots,k\}} \in A_{-\{1,\ldots,k\}}$. Then, $b_{\{1,\ldots,k\}}$ satisfies Condition 2, since (3) holds and $e_T(a_{-\{1,\ldots,k\}})$ is increasing with $a_{-\{1,\ldots,k\}}$. Thus, $\mathbf{g}^N$ is a nested pseudo-potential game. ■

Proof of Lemma 23. Suppose that $\mathbf{g}^N$ has a nested partition $\mathcal{T}$ pseudo-potential and the number of elements of $\mathcal{T}$ is L. Then, it is clear that we can find a sequence of partition potentials such that, at each step k, only two elements of the partition $\mathcal{T}^k$ are united, and the sequence of partitions reaches $\mathcal{T}$ from $\{i|i \in N\}$ in $n-L$ steps. That is, there exists a sequence $(\mathbf{f}^{\mathcal{T}^k})_{k=1}^{n-L}$ such that $(\mathcal{T}^k)_{k=1}^{n-L}$ is an unit increasingly coarser sequence of partitions of N; $\mathbf{f}^{\mathcal{T}^1} = \mathbf{g}^N$; $\mathbf{f}^{\mathcal{T}^{n-L}} = \mathbf{f}^{\mathcal{T}}$; and, for each $k = 2, 3, \ldots, n-L$, $\mathbf{f}^{\mathcal{T}^k}$ is a $\mathcal{T}^k$-pseudo-potential of $\mathbf{f}^{\mathcal{T}^{k-1}}$.

Fix any $k = 2, 3, \ldots, n-L$, any $j, l = 1, 2$ with $j \neq l$, and any $a_{-T_j} \in A_{-T_j^k}$. If $a^*_{T_j} \in \arg\max_{a_{T_j} \in A_{T_j}} f_{T_j^k}(a_{T_j}, a_{-T_j})$, then $a^*_{T_j}$ is a pure strategy Nash equilibrium of $\mathbf{g}^N|_{a_{-T_j^k}}$. That is, we can regard a best-response of T_j against a_{-T_j} as a pure strategy Nash equilibrium of $\mathbf{g}^N|_{-a_{T_j^k}}$. And, since, for each $k = 2, 3, \ldots, n-L$, $\mathbf{f}^{\mathcal{T}^{k-1}}$ has a $\mathcal{T}^k$-pseudo-potential, $\mathbf{f}^{\mathcal{T}^{k-1}}$ cannot have any strict best-response cycle as shown by Schipper (2004). Hence, by Lemma 12, we must find best-response selections $e_{T_1^k} : A_{-T_1^k} \to A_{T_1^k}$ and $e_{T_1^k} : A_{-T_2^k} \to A_{T_2^k}$ such that, for any $a_{-T_1^k \cup T_2^k} \in A_{-T_1^k \cup T_2^k}$ and any $a_j^k \in A_j^k$, $e_{T_j^k}(e_{T_l^k}(e_{T_j^k}(a_{T_l^k}, a_{-T_1^k \cup T_2^k}), a_{-T_1^k \cup T_2^k}), a_{-T_1^k \cup T_2^k}) \neq e_{T_j^k}(a_{T_l^k}, a_{-T_1^k \cup T_2^k})$. ■

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