# An "Invisible Hand" in Votes

Hirofumi Yamamura*

July 9, 2008, revised December 24, 2008

**Abstract**

We consider the problem of choosing a policy from a one-dimensional policy set in which voters have single-peaked preferences. The purpose of this paper is to analyze the consequences of strategic votes under any given voting rule satisfying some natural conditions; peak-only, unrestricted range, anonymity, continuity and weak (or strict) monotonicity. We show that, under any such rule, strategic votes must result in an outcome recommended by a generalized median voter rule. Our results suggest the existence of an "invisible hand" in votes through which strategic votes lead to a "reasonable" outcome.

JEL Classification: D78, D72, C70.
Key words: Generalized median voter rule, Average voting rule, Strategic manipulation, Existence of a strong Nash equilibrium, Coalition-proof Nash equilibrium, Uniqueness of a Nash equilibrium, Implementation.

## 1 Introcuction

We consider the problem of choosing a policy from a one-dimensional policy set in which voters have single-peaked preferences (Black [8]). In this environment, it is well known that there exist strategy-proof, efficient and anonymous voting rules, contrary to the Gibbard-Satterthwaite theorem (Gibbard [16]; Satterthwaite [34]). A typical example of such a rule is the median voter rule, which chooses the median of voters' peaks. Moreover, Moulin [28] provided a characterization of strategy-proof voting rules. He showed that a voting rule is strategy-proof, efficient, and anonymous if and only if it is a "generalized

*JSPS Research Fellow / Department of Social Engineering, Graduate School of Decision Science and Technology, Tokyo Institute of Technology, 2-12-1 Oh-okayama, Meguro-ku, Tokyo, 152-8552, Japan; yamamura-h@soc.titech.ac.jp

median voter rule," which chooses the median of $n$ voters' peaks and $n-1$ exogenous parameters. Because of the existence of "reasonable" and strategy-proof voting rules, many studies on this environment have focused on strategy-proof voting rules[1].

On the other hand, we have not paid much attention to what happens when a strategically manipulable voting rule is given. The purpose of this paper is to analyze the consequence of strategic votes under any given voting rule that satisfies some natural requirements[2]. The requirements we imposed on voting rules are peak-only, unrestricted range, anonymity, continuity and weak (or strict) monotonicity. Through the analysis of strategic votes under a given voting rule, we obtain the two results described below.

First, we show that under any voting rule that satisfies all of these properties, containing a strict monotonicity condition, the set of Nash, coalition-proof Nash and strong-Nash equilibrium outcomes must be equivalent to an outcome recommended by a particular generalized median voter rule (Theorem 1, (1).) Furthermore, we find that the set of Nash equilibria is "almost always" a singleton (Theorem 1, (3).) Second, we show that when a strict monotonicity condition is weakened to a weak monotonicity condition, the set of coalition-proof Nash and strong-Nash equilibrium outcomes will still be equivalent to a generalized median voter rule, while the set of Nash equilibrium outcomes no longer coincides with a generalized median voter rule (Theorem 2, (1).)

Renault and Trannoy [30] [31] analyzed the properties of the average voting rule and showed that the set of Nash and strong-Nash equilibrium outcomes under the average voting rule must be equivalent to a generalized median voter rule. This result can be obtained as a corollary of our Theorem 1, because the average voting rule satisfies the conditions on rules including strict monotonicity. Moreover, by applying our Theorem 2, we can still expect the consequence of strategic votes under a voting rule that satisfies weak monotonicity, but not strict monotonicity, for example, any generalized median voter rule.

Our two theorems reveal a strong position that generalized median voter rules possess. Even if we do not use a generalized median voter rule, voters' strategic votes result in a median voter rule, as if led by an "invisible hand (Smith [35])." Since a generalized median voter rule satisfies some desirable properties, such as efficiency and anonymity, this "invisible hand" has a role in ensuring a reasonable outcome. At the same time, our

[1] See for example, Barbera, Gul, and Stachetti [6], Ching [13], and so on. Sprumont [37], Barbera [5] and Jackson [19] offer surveys in this literature.

[2] There have been some studies that analyzes the consequence of strategic manipulations in several economic models. See, for example, Hurwicz [17] and Otani and Sicilian [29] in divisible goods economies, Tadenuma and Thomson [38] and Fujinaka and Sakai [15] in indivisible goods economies, Bochet and Sakai [9] in the division problem with single-peaked preferences.

theorems also have a negative aspect. They reveal that we cannot achieve any outcome, such as the average of voters' peaks, other than what a generalized median voter rule chooses. Whatever rule we use, we cannot escape from generalized median voter rules.

In the context of implementation theory, Theorem 1 shows that a strictly monotone voting rule can be regarded as a mechanism that implements a generalized median voter rule in Nash equilibria with two good properties; uniqueness of Nash equilibrium and robustness to coalitional deviations[3]. In particular, uniqueness of Nash equilibrium is an attractive property for Nash implementation. When there are multiple Nash equilibria as in many other Nash type mechanisms, it is difficult for voters to play a Nash equilibrium without communication even if voters expect the consequence of votes based on Nash equilibria, because voters cannot determine which Nash equilibrium is played without communication. In order to resolve such a problem, one solution is to construct a mechanism in which uniqueness of Nash equilibrium is assured. In a mechanism with a unique Nash equilibrium, Nash equilibrium can have a role to be a focal point among voters.

One might say we do not need to consider Nash implementation of a generalized median voter rule, because a strategy-proof rule, such as a generalized median voter rule, must induce sincere voting. However, experimental studies, such as Kawagoe and Mori [21], Attiyeh, Franciosi and Isaac [3], and Cason, Saijo, Sjostrom and Yamato [11] observe that strategy-proof mechanisms do not always work well because many subjects actually do not reveal true information. Saijo, Sjostrom and Yamato [32] point out that strategy-proof mechanisms sometimes fail to induce truth-telling because many of them have Nash equilibria that produce undesireble outcomes. To cope with such a fault of a strategy-proof mechanisms, they require double implementation in Nash and dominant strategy equilibrium called *secure implementation*. However, they also find that in the one-dimensional voting plobrem with single-peaked preferences, any generalized voter rule has Nash equilibria that cause inefficient outcomes. That is, there exists no efficient, anonymous and secure voting rule.

Thus, we have two ways to implement a generalized median voter rule. One way is to use a strategy-proof rule that has bad Nash equilibria. The other way is to use a rule that is not strategy-proof but can Nash implement the former. Our results imply that strictly monotone voting rules correspond to latter. If the problem of bad Nash equilibria of a generalized median voter rule is serious, a strictly monotone voting rules can be a

[3]There are some studies considering unique Nash implementation, such as Abreu and Sen [2], Abreu and Matsushima [1], Sjostrom [33], and Trockel [39]. However, the necessary and sufficient condition for unique Nash implementation has not been specified. This paper investigates a new environment in which a social choice rule can be implemented in a unique Nash equilibrium.

good alternative.

The rest of this paper is organized as follows: in section 2, we introduce the model and definitions; in section 3, we indicate our main result; in section 4, we interpret our results; and in section 5, we state our conclusions.

# 2 Notation

## 2.1 Basic Definitions

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of *voters.* $A \equiv [a, b]$ denotes the *policy set.* We assume that this policy set $A$ is a closed interval. For each voter $i \in N$, $i$ has a complete, transitive, and single-peaked preference $R_i$ over $A$. The symmetric and asymmetric parts of $R_i$ are denoted by $I_i$ and $P_i$ respectively. A preference relation $R_i$ over $A$ is said to be *single-peaked* if there exists a *peak* $p(R_i) \in [a, b]$ such that for each $c, d \in [a, b]$ $c < d \leq p(R_i)$ implies $dP_ic$ and $p(R_i) \leq c < d$ implies $cP_id$. Let $\mathcal{R}_i$ be the set of $i$'s single-peaked preferences[4].

A voting rule, denoted by $f$, is a function $f : \prod_{i \in N} \mathcal{R}_i \to A$ which associates with each preference profile $R \equiv \{R_i\}_{i \in N}$ a policy $f(R) \in A$.

## 2.2 "Peak-Only" Voting Rules and their Properties

Throughout this paper, we pay attention to *peak-only* voting rules. A voting rule $f$ is said to be *peak-only* if for any $R, R'$, $f(R) = f(R')$ whenever $p(R_i) = p(R'_i)$, for any $i \in N$. If a voting rule is peak-only, we can regard a voting rule $f : \prod_{i \in N} \mathcal{R}_i \to A$ as a function $f : [a.b]^N \to A$, that associates with each peak profile $p \in [a.b]^N$ a policy $f(p) \in A$.

In this paper, we additionally impose several natural conditions on peak only voting rules. They are listed as follows.

**Unrestricted Range**[5]: *For any $c \in A$, there exists $p \in [a, b]^N$ such that $f(p) = c$.*

**Anonymity**: *For any $p, p' \in [a, b]^N$, such that $\exists i, j \in N$ such that $p_i = p'_j$, $p_j = p'_i$ and $p_k = p'_k$, $\forall k \in N \setminus \{i, j\}$, $f(p) = f(p')$.*

**Own-Peak Continuity**: *For any $i \in N$, any $p_{-i} \in [a, b]^{N/\{i\}}$, and any $\{p_i^m\}_{m=1}^{\infty} \subseteq [a, b]$ such that $\lim_{m \to \infty} p_i^m = p_i$, $\lim_{m \to \infty} f(p_i^m, p_{-i}) = f(p_i, p_{-i})$.*

[4] Note that we do not assume continuity of $R_i$.

[5] Ching [13] calls this condition "voter sovereignty."

**Own-Peak Weak Monotonicity**: *For any* $p \in [a,b]^N$, *any* $i \in N$, *and any* $p'_i \geq p_i$, $f(p'_i, p_{-i}) \geq f(p)$.

**Own-Peak Strict Monotonicity**: *For any* $p \in [a,b]^N$, *any* $i \in N$, *and any* $p'_i > p_i$, $f(p'_i, p_{-i}) > f(p)$.

We illustrate some peak only voting rules that satisfy the conditions above.

**The median voter rule**: *Suppose that the number of voters is odd. For each* $R \in \prod_{i \in N} \mathcal{R}_i$, $f(R) = m(p(R_1), \cdots, p(R_n))$, *where* $m(x_1, \cdots, x_n)$ *denotes the median of* $x_1, \cdots, x_n$.

**Generalized Median Voter Rule**: *There exist* $a_1, \cdots, a_{n-1} \in [a.b]$, *such that for each* $R \in \prod_{i \in N} \mathcal{R}_i$, $f(R) = m(p(R_1), \cdots, p(R_n), a_1, \cdots, a_{n-1})$, *where* $m(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{2n-1})$ *denotes the median of* $x_1$, $\cdots$, $x_n$, $x_{n+1}$, $\cdots$, $x_{2n-1}$.

**The average voting rule**: *For each* $R \in \prod_{i \in N} \mathcal{R}_i$, $f(R) = \frac{\sum_{i \in N} p(R_i)}{n}$.

**Generalized average voting rule**: *There exists a continuous and strictly increasing function* $g : [na.nb] \to [a,b]$ *such that* $g(na) = a$, $g(nb) = b$, *and for each* $R \in \prod_{i \in N} \mathcal{R}_i$, $f(R) = g(\sum_{i \in N} p(R_i))$.

It can be easily checked that both generalized median voter rules and generalized average voting rules satisfy unrestricted range, anonymity, own-peak continuity, and own-peak weak monotonicity. On the other hand, generalized median voter rules do not satisfy own-peak strict monotonicity, while generalized average voting rules satisfy it.

These properties are similar to those Bochet and Sakai [9] imposed on division rules in the context of the division problem with single-peaked preferences. However, our properties are slightly different from them, because we replace efficiency[6], which Bochet and Sakai [9] imposed on division rules, with unrestricted range. It is clear that efficiency implies unrestricted range. We can obtain the same results even if efficiency is weakened to unrestricted range.

[6]In the problem of one-dimensional voting, we can easily be checked that a rule $f$ satisfies *efficiency* if and only if for any $p \in [a,b]^N$, there exists $i, j \in N$, such that $p(R_i) \leq f(p) \leq p(R_j)$.

## 2.3 Equilibrium Concepts

Under a voting rule $f$, each voter reports his own preference (or peak) and a policy is decided according to the reported preference (or peak) profile. Then, each voter will vote strategically so as to satisfy his own preference as much as possible. We expect the consequences of strategic votes based on Nash equilibrium or its refinements: coalition-proof Nash equilibrium and strong Nash equilibrium.

Given a peak-only rule $f : [a,b]^N \to A$ and a "true" preference profile $R \equiv \{R_i\}_{i \in I}$, let

$$N_v(f,R) \equiv \left\{ x \in [a,b]^N \mid \neg \left( \exists i \in N \, such \, that \, \exists x_i' \in [a,b] \, , \, f(x_i', x_{-i}) P_i f(x) \right) \right\}$$

be the *set of Nash equilibrium voting profiles* and

$$N(f,R) \equiv f(N_v(f,N))$$

be the *set of Nash outcomes* of a voting rule $f$ under a preference profile $R$.

We moreover introduce the concepts of *coalition-proof Nash equilibrium* (Bernheim, Peleg and Whinston [7]) and *strong Nash equilibrium* (Aumann [4]), which take coalitional deviations into consideration. When all voters can communicate with one another and deviate coalitionally, it may be more natural to expect the consequences based on strong or coalitional-proof Nash equilibria than based on Nash equilibria.

First, let us define a coalitional-proof Nash equilibrium. We say a coalition $S \subseteq N$ has a *credible deviation* $x_S' \in [a,b]^S$ at a voting profile $x$, if $f\left(x_S', x_{-S}\right) P_i f(x)$, $\forall i \in S$ and there is no $T \subseteq S$, $T \neq S$ such that $T$ has a credible deviation at $(x_S', x_{-S})$. A voting profile $x$ is a *coalition-proof Nash equilibrium* if there is no coalition that has a credible deviation at $x$. The *set of coalitional-proof Nash voting profiles* $CN_v(f,R)$ is defined as follows.

$$CN_v(f,R) \equiv \left\{ x \in [a,b]^N \mid \neg \left( \exists S \subseteq N \, such \, that \, S \, has \, a \, credible \, deviation \, at \, x \right) \right\}.$$

Let

$$CN(f,R) \equiv f(CN_v(f,N))$$

be the *set of coalitional-proof Nash outcomes.*

Next, let us define a *strong Nash equilibrium.* We say a coalition $S \subseteq N$ has a *deviation* $x_S' \in [a,b]^S$at a voting profile $x$, if $f\left(x_S', x_{-S}\right) P_i f(x)$, $\forall i \in S$. A voting profile $x$ is a *strong Nash equilibrium* if there is no coalition that has a deviation at $x$. Then, the set of strong Nash voting profiles $SN_v(f, R)$ is defined as

$$SN_v(f, R) \equiv \left\{x \in [a, b]^N \mid \neg\,(\exists S \subseteq N \, such\, that\, S\, has\, a\, deviation\, at\, x.)\right\},$$

and

$$SN(f, R) \equiv f(SN_v(f, R))$$

denotes the *set of strong Nash outcomes*.

By the definitions of these three equilibrium concepts, we can easily check that

$$SN(f, R) \subseteq CN(f, R) \subseteq N(f, R), \text{ for any } f \text{ and any } R.$$

## 3 Results

The purpose of this paper is to analyze the consequence of strategic votes under any given voting rule satisfying the properties introduced above. Before we proceed to our discussion, we introduce $n-1$ numbers that are useful for our analysis. Given a rule $f : [a, b]^N \to A$, for each $k \in \{1, 2, \cdots, n-1\}$, define $f_k \in [a, b]$ as

$$f_k \equiv f(\underbrace{b, \cdots, b}_{k}, \underbrace{a, \cdots, a}_{n-k}).$$

This $f_k$ indicates an outcome when $k$ voters vote for the upper bound of the policy set and the rest of voters vote for the lower bound. These $n-1$ numbers $f_1, \cdots, f_{n-1}$ help us analyze the consequence of strategic votes, because they actually play the roles of $n-1$ exogenous parameters of a generalized median voter rule.

Now, we provide our results of the analysis of strategic votes. Throughout this section, we assume without a loss of generality that $p(R_1) \le \cdots \le p(R_n)$.

**Proposition 1.** *Let $f : [a, b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak weak monotonicity. Then, for any $R \in \mathcal{R}$, there exists a voting profile $x \in SN_v(f, R)$ such that*

$$f(x) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}).$$

**Proof of Proposition 1**.
**Case 1**. $\{i \in N \mid p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \phi$.

Since $\{j \mid f_j = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} \neq \phi$ by the assumption and $f_1 \leq \cdots \leq f_j \leq \cdots \leq f_{n-1}$ by own-peak weak monotonicity of $f$, there exists $j \in \{1, \cdots, n-1\}$ such that

$$|\{i \in N \mid p(R_i) < f_j\}| + (j-1) = |\{i \in N \mid p(R_i) > f_j\}| + (n-1-j)^7.$$

Then, since $f_j = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$,we obtain

$$|\{i \in N \mid p(R_i) < f_j\}| = n - j \text{ and } |\{i \in N \mid p(R_i) > f_j\}| = j.$$

Let us consider the following voting profile $x \in [a,b]^N$ such that

$$x_i = a, \forall i \in \{1, \cdots, n-j\} \text{ and } x_i = b, \forall i \in \{n-j+1, \cdots, n\}.$$

By definition of $f_j$,

$$f(x) = f_j = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}).$$

We shall show that this voting profile $x \in [a,b]^N$ is a strong Nash equilibrium. Suppose there exist $S \subseteq N$ and $y_S \in [a,b]^S$such that $f(y_S, x_{-S}) P_i f(x)$, $\forall i \in S$. We moreover assume without a loss of generality that $f(y_S, x_{-S}) < f(x)$. Then,

$$S \subseteq \{1, \cdots, n-j\},$$

because $f(x) P_i f(y_S, x_{-S})$, $\forall i \in \{n-j+1, \cdots, n\}$. However, since $x_i = 0$, $\forall i \in S$,

$$f(y_S, x_{-S}) \geq f(x), \ \forall y_S \in [a,b]^S,$$

by own-peak weak monotonicity, which contradicts $f(y_S, x_{-S}) < f(x)$. Hence, $x \in [a,b]^N$ must be a strong Nash equilibrium.■

**Case 2**. $\{i \in N \mid p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} \neq \phi$.

By assumption, there exists $i \in N$ such that

$$|\{j \in \{1, \cdots, n-1\} \mid f_j \leq p(R_i)\}| + (i-1) = |\{j \in \{1, \cdots, n-1\} \mid f_j \geq p(R_i)\}| + (n-i).$$

[7] Given a finete set $X$, $|X|$ denotes the cardinality of $X$.

Then, since $p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$, we have

$$f_{n-i} \leq p(R_i) \leq f_{n-i+1}.$$

Let us consider the following voting profile $x_{-i} \in [a,b]^{N/\{i\}}$ such that

$$x_k = a, \forall k \in \{1, \cdots, i-1\}, \text{ and } x_k = b, \forall k \in \{i+1, \cdots, n\}.$$

Then, by own peak monotonicity, for any $x'_i \in [a,b]$

$$f(a, x_{-i}) = f_{n-i} \leq f(x'_i, x_{-i}) \leq f_{n-i+1} = f(b, x_{-i}),$$

so there exists $x_i \in [a,b]$ such that $f(x_i, x_{-i}) = p(R_i)$ by own peak continuity. We shall show that this voting profile $x = (x_i, x_{-i})$ is a strong Nash equilibrium.

Suppose there exist $S \subseteq N$ and $y_S \in [a,b]^S$ such that $f(y_S, x_{-S}) P_i f(x)$, $\forall i \in S$. We moreover suppose without loss of generality $f(y_S, x_{-S}) < f(x)$. Then,

$$S \subseteq \{1, , \cdots i-1\},$$

because $f(x) R_k f(y_S, x_{-S})$, $\forall k \in \{i, \cdots, n\}$. On the other hand, since $x_i = 0, \forall i \in S$,

$$f(y_S, x_{-S}) \geq f(x), \ \forall y_S \in [a,b]^S$$

by own-peak weak monotonicity, which contradicts $f(y_S, x_{-S}) < f(x)$. Hence, $x \in [a,b]^N$ must be a strong Nash equilibrium.■

Proposition 1 establishes the existence of a strong Nash equilibrium whose outcome corresponds to the median of $n$ voters' peaks and $n-1$ parameters defined as $f_1, \cdots$, and $f_{n-1}$. The proof of Proposistion 1 is constructive. First, focus on the outcome $m^* \in [a,b]$ which is the median of $p(R_1)$ ,$\cdots$ ,$p(R_n)$, $f_1$, $\cdots$, $f_{n-1}$. Next, construct a voting profile $x \in [a,b]^N$ such that

$$x_i = \begin{cases} a & \text{if } p(R_i) < m^* \\ x_i^* \text{ such that } f(x) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}) & \text{if } p(R_i) = m^* \\ b & \text{if } p(R_i) > m^* \end{cases}$$

and show that this voting profile $x$ is a Nash equilibrium

Note that in the literature of non-cooperative game theory considering coalitional deviations, Proposition 1 reveals a new sufficient condition for the existence of a strong Nash equilibrim. Sufficient conditions for the existence of a strong Nash equilibrium are provided by several studies, such as Ichiishi [18] and Konishi, Breton, and Weber [23]; however, we can apply none of them because our case does not satisfy any of them.

**Proposition 2.** *Let $f : [a,b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak weak monotonicity. Then, for any $R \in \mathcal{R}$, and any $x \in CN_v(f,R)$,*

$$f(x) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}).$$

**Proof of Proposition 2**. Suppose there exists $x \in CN_v(f,R)$ such that $f(x) > m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$. Let

$$S \equiv \{i \in N \mid p_i \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \{1, \cdots, i\} \text{ and}$$

$$T \equiv \{j \in \{1, \cdots, n-1\} \mid f_j \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \{1, \cdots, j\}.$$

Consider $x'_S \in [a,b]^S$ such that $x'_i = a, \forall i \in S$. Then, we have

$$f(x'_S, x_{-S}) \leq f_{n-i} \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}) < f(x),$$

because $|S|+|T| = i+j \geq n$, by the definition of $m$. Hence, there must exist $k \in \{1, \cdots, i\}$ such that

$$f(a, \cdots, a, x_k, \cdots, x_n) = f(x), \text{ and } f(a, \cdots, a, x_{k+1}, \cdots, x_n) < f(x).$$

Take $x''_k \in [a,b]$ such that

$$x''_k = \begin{cases} a & \text{if } f(a, \cdots, a, a, x_{k+1}, \cdots, x_n) > p(R_k) \\ c, \textit{ such that } f(a, \cdots, a, c, x_{k+1}, \cdots, x_n) = p(R_k) & \text{if } f(a, \cdots, a, a, x_{k+1}, \cdots, x_n) \leq p(R_k) \end{cases}$$

Then, since $p(P_1) \leq \cdots \leq p(R_k) \leq f(a, \cdots, a, x''_k, x_{k+1}, \cdots, x_n) < f(x)$,

$$f(a, \cdots, a, x''_k, x_{k+1}, \cdots, x_n) P_i f(x), \ \forall i \in \{1, \cdots, k\}.$$

Next, we show that $(a, \cdots, a, x''_k)$ is a credible deviation of $S' \equiv \{1, \cdots, k'\}$ at $x$. Let

$y \equiv (a, \cdots, a, x''_k . x_{-S'})$.

If $f(a, \cdots, a, x_{k+1}, \cdots, x_n) > p(R_k)$, then

$$f(y) \leq f(z_{S'}, y_{-S'}), \, \forall z_{S'} \in [a,b]^{S'}$$

by own-peak weak monotonicity, so

$$f(y) R_i f(z_{S'}, y_{-S'}), \, \forall S' \subseteq S, \, \forall y_{S'} \in [a,b]^{S'}, \, \forall i \in S'$$

Hence, there exists no $S'' \subset S'$ that has a credible deviation at $y$. Thus, $(a, \cdots, a, x''_k)$ must be a credible deviation of $S' \equiv \{1, \cdots, k\}$at $x$ which contradicts $x \in CN_v(f, R)$.

If $f(a, \cdots, a, x_{k+1}, \cdots, x_n) \leq p(R_k)$, then

$$f(y) R_k f(z_{S'}, x_S), \, \forall z_{S'} \in [a,b]^{S'},$$

because $f(y) = p(R_k)$. Hence, if there exists $S'' \subset S'$ that has a credible deviation at $y$, then $k \notin S''$. However, by own-peak weak monotonicity,

$$f(z_{\{1,\cdots,k-1\}}, y_{\{k,\cdots,n\}}) \geq f(y), \; \forall z_{\{1,\cdots,k-1\}} \in [a,b]^{\{1,\cdots,k-1\}},$$

which implies

$$f(y) R_i f(z_{\{1,\cdots,k-1\}}, y_{\{k,\cdots,n\}}), \, \forall z_{\{1,\cdots,k-1\}} \in [a,b]^{\{1,\cdots,k-1\}}$$

Hence, there exists no $S'' \subset S'$that has a credible deviation at $y$. Thus, $(a, \cdots, a, x''_k)$ must be a credible deviation of $S' \equiv \{1, \cdots, k\}$ at $x$, which contradicts $x \in CN_v(f, R)$. Therefore, if $x \in CN_v(f, R)$, then $f(x) \leq m(p_1, \cdots, p_n, f_1, \cdots, f_{n-1})$.

We can similarly show that if $x \in CN_v(f, R)$, then $f(x) \geq m(p_1, \cdots, p_n, f_1, \cdots, f_{n-1})$.■

Proposition 1 and Proposition 2 together chatacterize coalitional proof Nash and strong Nash outcomes. As a direct consequence of Proposition 1 and Proposition 2, we can state that if a voting rule satisfies unrestricted range, anonymity, own-peak continuity, and own-peak weak monotonicity, then

$$SN(f, R) = CN(f, R) = \{m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\},$$

which implies that strategic votes must result in $m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$ when communication among voters are allowed.

As a next step, we check whether any Nash outcome is equivalent to $m(p(R_1), \cdots, p(R_n),$

$f_1, \cdots, f_{n-1})$ or not. Generally speaking, this equivalence does not hold. The following Example 1 establises the difference between Nash outcomes and $m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$.

**Example 1.** Let $N \equiv \{1, 2, 3\}$ and $f(x) = m(x_1, x_2, x_3)$. Then, for any $x \in A$, $x \equiv (x, x, x)$ is always a Nash equiribrium, because for any $i \in \{1, 2, 3\}$ and any $x'_i \in A$, $m(x'_i, x_{-i}) = x$. Therefore, for any $R \in \mathcal{R}$, $N(f, R) = A$.■

In order to establish the equivalence between Nash outcomes and $m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$, we have to impose own-peak strict monotonicity on voting rules additionally.

**Proposition 3.** *Let $f : [a, b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak strict monotonicity. Then, for any $R \in \mathcal{R}$, any $x \in N(f, R)$*

$$f(x) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}).$$

**Proof of Proposition 3**. Suppose there exists $x \in N_v(f, R)$ such that $f(x) > m(p_1, \cdots, p_n, f_1, \cdots, f_{n-1})$. Let

$$S \equiv \{i \in N \mid p(R_i) \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \{1, \cdots, i\}, \text{ and}$$

$$T \equiv \{j \in \{1, \cdots, n-1\} \mid f_j \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \{1, \cdots, j\}.$$

First, let us show that $x_i = a$, $\forall i \in S$ if $f(x) > m(p_1, \cdots, p_n, f_1, \cdots, f_{n-1})$ and $x \in N_v(f, R)$. Suppose $x_i > a$ $\exists i \in S$. Then, there exists $x'_i \in (a, x_i)$ such that

$$p(R_i) < f(x'_i, x_{-i}) < f(x),$$

by own-peak strict monotonicity and own-peak continuity. Hence,

$$f(x'_i, x_{-i}) P_i f(x),$$

which contradicts $x \in N_v(f, R)$.

However, if $x_i = a \ \forall i \in S$,

$$f(x) \leq f_{n-i} \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}), \ \forall x_{N \setminus S} \in [a,b]^{N \setminus S}$$

because $|S|+|T| = i+j \geq n$, which contradicts $f(x) > m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$. Hence, we obtain for any $x \in N_v(f, R)$,

$$f(x) \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}),$$

We can similarly show that for any $x \in N_v(f, R)$,

$$f(x) \geq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}).\blacksquare$$

Combining Proposition 3 with Proposition 1, we establish the equivalence among Nash outcomes, coalition-proof Nash outcomes and the strong Nash outcomes if a voting rule additionally satisfies own-peak strict monotonicity. They together imply that if a voting rule satisfies unrestricted range, anonymity, own-peak continuity, and own-peak strict monotonicity, then

$$SN(f, R) = CN(f, R) = N(f, R) = \{m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\}.$$

Hence, under a strictly monotone voting rule, strategic votes must result in $m(p(R_1), \cdots,$ $p(R_n),\ f_1, \cdots, f_{n-1})$, whether voters is allowed to communicate with one another or not.

**Proposition 4.** *Let $f : [a,b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak weak monotonicity. Then, for any $R \in \mathcal{R}$,*

$$SN_v(f, R) = CN_v(f, R).$$

**Proof of Proposition 4**. It is sufficient to show that there must exist $S' \subseteq N$ which has a credible deviation at $x \in [a,b]^N$, whenever there exists $S \subseteq N$ which has a deviation at $x \in [a,b]^N$. Suppose that $S \subseteq N$ has a deviation $y \in [a,b]^S$at $x \in [a,b]^N$. We moreover assume without a loss of generality $f(y_S, x_{-S}) < f(x)$. Then, for any $i \in S$, $p(R_i) < f(x)$. Let $S \equiv \{i_1, \cdots, i_k\}$ such that $p_{i_1} \leq \cdots \leq p_{i_k}$. Consider $y_S \in [a,b]^S$, such that $y_i = a$,

$\forall i \in S$. Then, there must exist $h \in \{1, \cdots, k\}$such that

$$f\left((y_i)_{i\in\{i_1,\cdots,i_{h-1}\}}, (x_i)_{i\in N/\{i_1,\cdots,i_{h-1}\}}\right) = f(x), \text{ and}$$

$$f\left((y_i)_{i\in\{i_1,\cdots,i_h\}}, (x_i)_{i\in N/\{i_1,\cdots,i_h\}}\right) < f(x)$$

Let $S' \equiv \{i_1, \cdots, i_h\}$ and take $(z_i)_{i\in S'}$ such that $z_i = a$, $\forall i \in \{i_1, \cdots, i_{h-1}\}$, and

$$z_{i_h} = \begin{cases} a & \text{if } f\left(\{z_i\}_{i\in S'/\{i_h\}}, a, \{x_i\}_{i\in N/S'}\right) > p(R_{i_h}) \\ c, \textit{ such that } f\left(\{z_i\}_{i\in S'/\{i_h\}}, c, \{x_i\}_{i\in N/S'}\right) = p(R_{i_h}) & \text{if } f\left(\{z_i\}_{i\in S'/\{i_h\}}, a, \{x_i\}_{i\in N/S'}\right) \leq p(R_{i_h}) \end{cases}$$

We can show that $\{z_i\}_{i\in S'}$is a credible deviation of $S'$ at $x$ in a similar way with the proof of Proposition 2.■

Proposition 4 states a voting profile $x$ is a strong Nash equilibrium whenever $x$ is a coalition-proof Nash equilibrium. Note that by Proposition 1 and Proposition 4 we find a new sufficient condition for the equivalence between strong Nash and coalition-proof Nash equilibria. Konishi, Breton, and Weber [24] showed sufficient conditions for the equivalence between these two solutions, but we cannot apply them to our case.

**Proposition 5.** *Let $f : [a,b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak strict monotonicity. Then, for any $R \in \mathcal{R}$, any $x \in N(f,R)$*

$$SN_v(f,R) = CN_v(f,R) = N_v(f,R).$$

**Proof of Proposition 5**. It suffices to show that $x \notin N_v(f,R)$ whenever $x \notin SN_v(f,R)$. Suppose that $S \subseteq N$ has a deviation $y \in [a,b]^S$at $x \in [a,b]^N$. We moreover assume without a loss of generality $f(y_S, x_{-S}) < f(x)$. Then, by own-peak strict monotonicity, there exists $i \in S$ such that $y_i < x_i$ and $P(R_i) < f(x)$. By own-peak strict monotonicity and own-peak continuity, there exists $z_i \in (y_i, x_i)$ such that $f(z_i, x_{-i}) \in (P(R_i), f(x))$. Hence, $f(z_i, x_{-i}) P_i f(x)$, that implies $x \notin N_v(f,R)$.■

Proposition 5 states that the set of Nash equilibria is also equivalent to that of strong Nash and coalition-proof Nash equilibria if a voting rule additionally satisfies own-peak strict monotonicity.

**Proposition 6.** *Let $f : [a,b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak strict monotonicity. Then, for any $R \in \mathcal{R}$, such that $|\{i \in N \mid p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\}| \leq 1$, $N_v(f, R)$ is a singleton.*

**Proof of Proposition 6**. Since non-emptiness of $N_v(f, R)$ has been proved in Proposition 1, it suffices to show that $N_v(f, R)$ is a singleton.

**Case 1**. $\{i \in N \mid p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \phi$.

Let $S \equiv \{i \in N \mid p(R_i) < m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\} = \{1, \cdots, i\}$. If $x \in N_v(f, R)$, then by Proposition 4,

$$f(x) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}).$$

If $x \in N_v(f, R)$, then we must have $x_i = a$, $\forall i \in S$, because for any $x_i' \neq a$,

$$p(R_i) < f(x) < f(x_i', x_{-i}),$$

which implies that $f(x) P_i f(x_i', x_{-i})$.

Similarly, we have $x_i = b$, $\forall i \in N/S$. Thus, if $x \in N_v(f, R)$, then $x_i = a$, $\forall i \in S$ and $x_i = b$, $\forall i \in N/S$.$\blacksquare$.

**Case 2.** $|\{i \in N \mid p_i = m(p_1, \cdots, p_n, f_1, \cdots, f_{n-1})\}| = 1$

Take $i \in N$ such that $p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$. Then, if $x \in N_v(f, R)$, we must have

$$x_i = a, \forall k \in \{1, \cdots, i-1\},$$

because $p(R_k) < f(x) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})$, $\forall k \in \{1, \cdots, i-1\}$. Similarly, if $x \in N_v(f, R)$,

$$x_k = b, \forall k \in \{i+1, \cdots, n\}.$$

Thus, we obtain that if $x \in N_v(f, R)$, then

$$f(x_{-i}, a) = f_{n-i} \leq f(x) \leq f_{n-i+1} = f(x_{-i}, b).$$

By own-peak strict monotonicity and own-peak continuity, there exists a unique $\bar{x}_i \in [a, b]$ such that

$$f(\bar{x}_i, x_{-i}) = p_i \in [f_{n-i}, f_{n-i+1}].$$

Since $f(x'_i, x_{-i}) \neq p_i$, for any $x'_i \neq \bar{x}_i$, $f(\bar{x}_i, x_{-i}) P_i f(x'_i, x_{-i})$, which indicates that $x_i = \bar{x}_i$, if $x \in N_v(f, R)$. Therefore, if $x \in N_v(f, R)$, then

$$x = (a, \cdots, a, \bar{x}_i, b, \cdots, b).\blacksquare$$

Proposition 6 ensures uniqueness of Nash equilibrium under a strictly monotone voting rule. Since the case $|\{i \in N \mid p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\}| \geq 2$ holds are non-generic, the set of Nash equilibrium voting profiles is "almost always" a singleton. However, there may be multiple Nash equilibria if there are not less than 2 persons whose peaks agree with the median of $p(R_1), \cdots, p(R_n)$, $f_1, \cdots, f_{n-1}$. The following Example 2 illustrates the case in which there are multiple Nash equilibria.

**Example 2.** Let $N \equiv \{1, 2, 3\}$, $p(R_1) = p(R_2) = p(R_3) = \frac{a+b}{2}$ and $f(x) = \frac{x_1+x_2+x_3}{3}$. Then, we can easily check that for any $x$ such that $\frac{x_1+x_2+x_3}{3} = \frac{a+b}{2}$, $x$ is a Nash equilibrium.$\blacksquare$

Now, we summarize our analysis on the consequence of strategic votes as the following Therem 1 and Theorem 2.

**Theorem 1.** *Let $f : [a, b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak strict monotonicity. Then, for any $R \in \mathcal{R}$, the following* (1), (2), *and* (3) *hold.*
(1) $SN(f, R) = CN(f, R) = N(f, R) = \{m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\}$.
(2) $SN_v(f, R) = CN_v(f, R) = N_v(f, R) \neq \phi$.
(3) $N_v(f, R)$ *is a singleton if* $|\{i \in N \mid p(R_i) = m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\}| \leq 1$.

**Proof of Theorem 1**. (1) is implied by Proposition 1 and Proposition 3. (2) is implied by Proposition 1 and Proposition 5. Proposition 6 directly implies (3).$\blacksquare$

**Theorem 2.** *Let $f : [a, b]^N \to A$ be any voting rule that satisfies unrestricted range, anonymity, own-peak continuity, and own-peak weak monotonicity. Then, for any $R \in \mathcal{R}$, the following* (1) *and* (2) *hold.*
(1) $SN(f, R) = CN(f, R) = \{m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1})\}$.

(2) $SN_v(f,R) = CN_v(f,R) \neq \phi$.

**Proof of Theorem 2**. (1) is implied by Proposition 1 and Proposition 2. (2) is implied by Proposition 1 and Proposition 4.■

# 4 Discussions

## 4.1 Implementation of Generalized Median Voter Rules

Theorem 1 reveals that any genaralized median voter rule $m(p(R_1, \cdots, p(R_n), a_1, \cdots, a_{n-1})$, such that $a < a_1 < \cdots < a_{n-1} < b$ can be triply implemented in strong Nash, coalition-proof Nash, and Nash equilibria by a strictly monotone voting rule. For example, let us use a generalized average voting rule $f(x) = g(\sum_{i \in N} x_i)$ such that $g((n-k)a + kb) = a_k$, for each $k \in \{1, \cdots, n-1\}$. Then, by Theorem 1, any Nash, coalition-proof Nash, and strong Nash equilibrium of this generalized median rule must result in the median of $p(R_1), \cdots, p(R_n), a_1, \cdots, a_{n-1}$. Moreover, a strictly monotone rule has attractive properties as a Nash mechanism: uniqueness of Nash equilibrium and robustness to coalitional deviations. When uniqueness of Nash equilibrium is assured, the unique equilibrium can have a role of a focal point among voters. When a mechanism can triply implementation a social choice rule in strong Nash, coalition-proof Nash, and Nash equilibria, social planners need not be worried about coalitional deviations, as well as individual deviations.

In addition, Kawasaki and Yamamura [22] find out that the uniquie Nash equilibrium under a generalized average voting rule is dynamically stable in the sense similar to Cournot stability, by using the theory of *potential games*[8]. It is known that under a potential game, any best response path converges to Nash equilibria (Monderer and Shapley [27]; Jensen [20]). They show that a voting situation under a generalized average voting rule is a kind of a potential game and so that stability of Nash equilibrium is assured. They also point out that generalized median voter rules do not have this property.

One might think that we do not have to study Nash implementation of generalized median voter rules because they must induce truth-telling. However, this expectation is denied by experimental studies, such as Kawagoe and Mori [21], Attiyeh, Franciosi and

[8]Note that the class of generalized average voting rules is a subclass of strictly monotone voting rules. Kawasaki and Yamamura [22] showed only stability of generalized average voting rules. However, it can be easily checked that the class of rules that can be Nash implemented by a generalized average voting rule is equivalent to the class of rules that can be Nash implemented by a strictly monotone voting rule.

Isaac [3], and Cason, Saijo, Sjostrom and Yamato [11]. They observe that few subjects reveal true information under some strategy-proof mechanisms.

Saijo, Sjostrom and Yamato [32] point out that this is because many strategy-proof mechanisms has a drawback of having Nash equilibria that cause undesireble outcomes. They also show that any generalized median voter rule has this drawback. For example, as seen in Example 1, the set of Nash outcomes under the median voter rule covers the entire policy set. This means that a direct revelation mechanism of the median voter rule fails to implement itself in Nash equilibria.

Confronting such a fault of a direct revelation mechanism of a generalized median voter rule, we have two ways to implement it. One way is to keep using a direct revelation mechanism. Though it has bad Nash equilibria that cause undesirable outcomes, it implements itself in that dominant strategy equilibria. The other way is to use a strictly monotone voting rule. Though it is not strategy-proof, it can exclude bad Nash equilibria. Especially, a generalized average voting rule ensures that voters learn to play a Nash equilibrium. We need to explore laboratory experiments to observe how serious the problem of bad Nash equibibria of a generalized median voter is and which works better[9].

The comparison among rules mentioned here is summarized by the following table 1.

Table 1: Comparison of Voting Rules

| | Generalized Median Rules | Generalized Average Rules | Strictly Monotone Rules |
|---|---|---|---|
| Strategy-proofness | *yes* | *no* | *no* |
| Nash Implemantation | *no* | *yes* | *yes* |
| Uniqueness of NE | *no* | *yes* | *yes* |
| Dynamic Stability of NE | *no* | *yes* | |
| Robustness to Coalitional Deviations | *yes* | *yes* | *yes* |

## 4.2 Relation to Dasgupta-Hammond-Maskin Theorem

Dasguputa, Hammond, and Maskin [14] proved that a single-valued social choice rule must be strategy-proof if and only if it satisfies Maskin monotonicity, a necessary condi-

[9] Bochet, Saijo, Sakai, Yamamura and Yamato [10] conduct an experiment for the division problem with single-peaked preferences to compare the performance of the uniform rule with that of the proportional rule that Nash implements the uniform rule. They observe that the proportional rule sometimes yields the outcomes closer to the uniform allocation than the uniform rule. They report this finding in an incomplete information setting in which each subject is imformed of only his own payoff function.

tion for Nash implementation (Maskin [25] [26].) Hence, it is not unnatural to regard our results as a corollary of Dasgupta, Hammond, and Maskin [14] and Moulin [28], because any Nash implementable and single-valued social choice rule must be strategy-proof. However, this intuition is not correct for the reasons given below.

First, the unique existence of Nash equilibrium outcomes in a strict monotone rule is not derived from Dasgupta, Hammond and Maskin [14]. Second, even if there exists a unique Nash equilibrium outcome, Dasgupta-Hammond-Maskin theorem cannot conclude that this unique outcome is efficient or anonymous, because there are some strategy-proof rules that are not efficient and anonymous[10].

### 4.3 Comparison with Bochet and Sakai [9]

Bochet and Sakai [9] analyzed the consequence of strategic manipulations in the division problem with single-peaked preferences under a given division rule and obtained the following two results.
(1) Under any division rule satisfying own-peak strict monotonicity, efficiency, and some supplementary conditions, the set of strong Nash, efficient Nash, and Nash equilibrium outcomes coincides with the uniform rule, which is the unique division rule satisfying strategy-proofness, efficiency, and anonymity (Sprumont [36]; Ching [12].)
(2) When own-peak strict monotonicity is weakened to own-peak weak monotonicity, the set of strong Nash and efficient Nash equilibrium outcomes still coincides with the uniform rule.

Our results are similar to those of Bochet and Sakai [9], because both their paper and this paper show the equivalence between the consequence of strategic manipulations and efficient, anonymous, and strategy-proof rules. However, our results are slightly different from theirs. The differences are summarized as follows:

First, we have provided a sufficient condition for uniqueness of Nash equilibria, while they did not. Finding mechanisms that "almost always" has a unique Nash equilibrium is potentially one of our main findings. Second, we prove the general equivalence between the set of coalition-proof Nash equilibrium outcomes and a generalized median voter rule. Though they also suggest this equivalence, they leave this question open. Third, as stated in section 2.2, they impose efficiency on division rules, while we impose only unrestricted range instead of efficiency on voting rules.

On the other hand, we cannot show the equivalence between the set of efficient Nash equilibrium outcomes and a generalized median voter rule. As seen in example introduced

[10] For example, consider voting rules $f$ and $g$ such that $f(x) = x_i$, $g(x) = c$, for each $x \in [a, b]^N$ . Clearly both $f$ and $g$ are strategy-proof, but $f$ fails anonymity and $g$ fails efficiency.

in Section 4.1, a weakly monotone voting rule can have efficient Nash equilibria what causes outcomes other than what a generalized median voter rule chooses.

# 5 Concluding Remarks

Through the analysis of strategic votes, we reveal the strong position that generalized median voter rules possess. They are not only strategy-proof but also always expected as an outcome of strategic manipulation under any given voting rule satisfying mild conditions. Since generalized median voter rules are efficient and anonymous, if only efficiency and anonymity matter, we can conclude that individual voting activities lead to an ethically desirable outcome through an "invisible hand." However, this result also has a negative aspect. This "invisible hand" makes us unable to escape from a generalized median voter rule, whatever rule we use.

In the context of implementation theory, we find out that any generalized median voter rule $m(p(R_1), \cdots, p(R_n), a_1, \cdots, a_{n-1})$,such that $a < a_1 < \cdots < a_{n-1} < b$, can be Nash implemented by a strictly monotone voting rule that has good properties: uniqueness of Nash equilibrium and robustness to coalitional deviations. However, this mechanism does not have a dominant strategy as in the direct revelation mechanism of a generalized median voter rule. On the other hand, as pointed by Saijo, Sjostrom and Yamato [32], since a direct revelation mechanism of a generalized median voter rule has "bad" Nash equilibria that cause undesirable outcomes, it does not implement itself in Nash equilibria.

As shown by Saijo, Sjostrom and Yamato [32], any generalized median voter rule cannot be doubly implemented in both Nash and dominant strategy equilibria. Then, in order to implement it, we should use either a Nash mechanism or a strategy-proof mechanism as a second best way. It is theoretically and empirically unclear which mechanism works better to implement a generalized median voter rule. To study this issue, we need to explore laboratory experiments as done by Cason, Saijo, Sjostrom and Yamato [11] and Bochet, Saijo, Sakai, Yamamura and Yamato [10].

**Acknowledgement.** I am grateful to Herve Moulin, Tatsuyoshi Saijo, Tomas Sjostrom, John Weymark and Takehiko Yamato for their valuable advices and discussions. I also thank participants of the 9th International Meeting of the Society for Social Choice and Welfare at Concordia University, the 27th Arne Ryde Symposium at Lund University, and seminars at UCSD, Tokyo Institute of Technology and Waseda University for helpful comments. This research is financially supported by the Society for Promotion of Science under Grand-in-Aid for JSPS Fellows 20009834, which is gratefully acknowledged.

## References

[1] Abreu, D. and Matsushima, H. (1992) "Virtual Implementation in Iteratively Undominated Strategies: Complete Information," Econometrica, 60, pp. 993-1008.

[2] Abreu, D. and Sen, A. (1991) "Virtual Implementation in Nash Equilibrium," Econometrica, 59, pp. 997-1021.

[3] Attiyeh, G., Franciosi, R. and Isaac, R.M. (2000) "Experiments with the pivotal process for providing public goods," Public Choice, 102, pp. 95-114.

[4] Aumann, R. (1959) "Acceptable Points in General Cooperative n-Person Games," in Kuhn, H.W., and Luce, R.D., eds. Contribution to the Theory of Games, pp.287-324, Princeton University Press.

[5] Barbera, S. (2001) "An Introduction to Strategy-Proof Social Choice Functions," Social Choice and Welfare, 18, 619-653.

[6] Barbera, S., Gul, F., and Stachetti, E. (1994) "Generalized Median Voter Schemes and Committees," Journal of Economic Theory, 61, 262-89.

[7] Bernheim, D., Peleg, B., and Whinston, M. (1987) "Coalition-Proof Nash Equilibria I. Concepts," Journal of Economic Theory, 42, 1-12.

[8] Black, D. (1948) "On the Rationale of Group Decision Making," Journal of Political Economy, 56, pp. 23-34.

[9] Bochet, O. and Sakai, T. (2007) "Preference Manipulations Lead to the Uniform Rule," mimeo.

[10] Bochet, O., Saijo, T., Sakai, T., Yamamura, H. and Yamato, T. (2008) "Manipulation of Preferences in Rationing Problems: Experimental Evidence," mimeo.

[11] Cason, T., Saijo, T., Sjostrom, T., and Yamato, T. (2006) "Secure Implementation Experiments: Do Strategy-Proof Mechanisms Really Work?," Games and Economic Behavior, 57, pp. 206-235.

[12] Ching, S. (1994) "An Alternative Characterization of the Uniform Rule," Social Choice and Welfare, 40, pp. 57-60.

[13] Ching, S. (1997) "Strategy-Proofness and 'median voters,"' International Journal of Game Theory, 26, pp. 473-490.

[14] Dasgupta, P., Hammond, P., and Maskin, E. (1979) “The Implementation of Social Choice Rules: Some General Results on Incentive Compatibility,” Review of Economic Studies, 46, pp. 185-216.

[15] Fujinaka, Y. and Sakai, T. (2007). "The manipulability of fair solutions in assignment of an indivisible object with monetary transfers," Jourlal of Public Economic Theory, 9, pp. 993-1011.

[16] Gibbard, A. (1973) “Manipulation of Voting Schemes: A General Result,” Econometrica, 41, pp. 587-601.

[17] Hurwicz, L. (1978). “On the Interaction between Information and Incentives in Organization,” in Communications and Interactions in Society (Eds. by K. Krippendorf), New York, Scientific Publishers.

[18] Ichiishi, T. (1993) “The Cooperative Nature of the Firm,” Cambridge University Press.

[19] Jackson, M.O. (2003) “Mechanism Theory,” in *Optimization and Operations Research*, (Eds. by U. Derigs), in the Encyclopedia of Life Support Systems, EOLSS Publishers.

[20] Jensen, M. K. (2007) "Stability of Pure Strategy Nash Equilibrium in Best-reply Potential Games," mimeo.

[21] Kawagoe, T., and Mori, T. (2001). "Can the pivotal mechanism induce truth-telling? An experimental study," Public Choice 108, pp. 331-354.

[22] Kawasaki, R. and Yamamura, H. (2008) "Poteitial Mechanisms in Single-Peaked Environments: Average Rules and Proportional Rules," mimeo.

[23] Konishi, H., Breton, M., and Weber, S. (1997a) “Equilibria in a Model with Partial Rivalry,” Journal of Economic Theory, 72, pp. 225-237.

[24] Konishi, H., Breton, M., and Weber, S. (1997b) “Equivalence of Strong and Coalition- Proof Nash Equilibria in Games without Spillovers,” Economic Theory, 9, pp. 97-113.

[25] Maskin, E. (1979) “Incentive Schemes Immune to Group Manipulation,” mimeo.

[26] Maskin, E. (1999) “Nash Equilibrium and Welfare Optimality,” Review of Economic Studies, 66, pp. 23-38.

[27] Monderer, D. and Shapley, L. S., 1996. Potential Games. Games and Economic Behavior, 14, pp. 124-143.

[28] Moulin, H. (1980) "On Strategy-Proofness and Single Peakedness," Public Choice, 35, pp. 437-455.

[29] Otani, Y. and Sicilian, J. (1982). "Equilibrium and Walras Preference Game," Journal of Economic Theory, 29, pp. 365-374.

[30] Renault, R. and Trannoy, A. (2004) "Assesing the Extent of Strategic Manipulations for the Average Voting Rule," mimeo.

[31] Renault, R. and Trannoy, A. (2005) "Protecting Minorities through the Average Voting Rule," Journal of Public Economic Theory, 7, pp. 169-199.

[32] Saijo, T., Sjostrom, T., and Yamato, T. (2007) "Secure Implementaion," Theorecial Economics, 2, pp. 203-229.

[33] Sjostrom, T. (1994) "Implementation in Undominated Nash Equilibria without Integer Games," Games and Economic Behavior, 6, pp. 502-511.

[34] Satterthweite, M.A. (1975) "Strategy-Proofness and Arrow's Conditions: Existence and Corresponding Theorems for Voting Procedures and Social Welfare Functions," Journal of Economonic Theory, 10, pp. 187-217.

[35] Smith, A. (1776) "An Inquiry into the Nature and Causes of the Wealth of Nations," University of Chicago Press.

[36] Sprumont, Y. (1991) "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule," Econometrica, 59, pp. 509-519.

[37] Sprumont, Y. (1995) "Strategy-Proof Collective Choice in Economic and Political Enviornments," Canadian Journal of Economics, 28, pp. 68-107.

[38] Tadenuma, K. and Thomson, W. (1995). "Games of Faie Division," Games and Economic Behavior 9, pp. 191-204.

[39] Trockel, T. (1999) "Unique Nash Implementation for a Class of Bargaining Solutions," International Game Theory Review, 1, pp. 267-272.