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Strategy-Proof Rule in Probabilistic Allocation Problem of an Indivisible Good and Money

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Strategy-Proof Rule in Probabilistic Allocation Problem of an Indivisible Good and Money

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Abstract

We consider the problem of probabilistically allocating a single indivisible good among agents when monetary transfers are allowed. We construct a new strategy-proof rule, called the second price trading rule, and show that it is second best efficient. Furthermore, we give the second price trading rule three characterizations with (1) strategy-proofness, "budget-balance", equal treatment of equals, weak decision-efficiency, and simple generatability, (2) strategy-proofness, "equal rights lower bound", equal treatment of equals, weak decision-efficiency, and simple generatability, (3) strategy-proofness, "envy-freeness, no-trade-no-transfer", equal treatment of equals, weak decision-efficiency, and simple generatability.

Keywords: Strategy-proofness, Probabilistic allocation problem, Second price trading rule, Budget-balance, Second best efficiency

JEL Classification numbers: D71, D78

1 Introduction

We study the probabilistic allocation problem of a single indivisible good among agents when monetary compensations are possible. Each agent has a preference expressed by quasi-linear utility function and maximizes his expected utility. A rule determines an assignment probability of the indivisible good and a monetary transfer to each agent for each preference profile. We

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consider incentive-compatible rules that elicit the preference of each agent. We especially focus on *strategy-proof* rule, where the truthful report of one's preference is always dominant strategy.

On the deterministic allocation problems, many researches have studied the class of the Groves rules, which is the only class of rules satisfying strategy-proofness and *decision-efficiency* [Holmström (1979)]. Decision-efficiency requires that the good is assigned to an agent who has the highest value. Among them, the Vickery (1961) rule[1] is one of the most analyzed rules. It is well-known that the Vickery rule is the only deterministic rule satisfying strategy-proofness, *individual rationality*, *non-positive transfer*, and either decision-efficiency [Holmström (1979)], *envy-freeness* [Svensson (1983)], or *equal treatment of equals*[2] (*anonymity*)[3] [Ashlagi and Serizawa (2012)]. Individual rationality requires that no one be worse off than the initial state. Non-positive transfer requires that any agent's transfer be non-positive. Envy-freeness requires that no one prefer other agent's assignment to his own. Equal treatment of equals requires that the agents who have the same preference be treated equally. Anonymity requires that a rule be defined independently of the names of the agents.[4]

Although the Vickery rule has excellent features, it has also several drawbacks. A well-known drawback is that the Vickery rule does not satisfy budget-balance. Budget-balance requires that the total amount of monetary transfers is always zero. This implies that monetary transfers flow out of agents. This drawback is, however, not particular to the Vickery rule, because all the Groves rules do not satisfy budget-balance [Green and Laffont (1977)]. Furthermore, even if we consider finitely restricted domains, there exists no deterministic rule satisfying strategy-proofness, budget-balance, and[5] neither decision-efficiency [Ohseto (2000)], envy-freeness [Ohseto (2000)], nor equal treatment of equals [Kato et al. (2015)]. Hence, it is very difficult to achieve budget-balance among deterministic rules.[6]

To improve welfares of agents but not achieving budget-balance, the Bai-

[1]Sometimes, this rule is called VCG [Vickery (1961), Clarke (1971), and Groves (1973)] rule.

[2]Strictly speaking, the results on the deterministic model are valid with equal treatment of equals in welfare or anonymity in welfare.

[3]See also Sakai (2013).

[4]Anonymity implies equal treatment of equals.

[5]Ando et al. (2008) have constructed a rule satisfying strategy-proofness, budget-balance, individual rationality, and weak symmetry on heavily restricted domain.

[6]Fujinaka (2008) has designed an outstanding rule, which satisfies Bayesian incentive compatibility, individual rationality, budget-balance, decision-efficiency, envy-freeness, and anonymity, but does not strategy-proofness.

ley (1997) rule has been recently paid attention by many researchers.[7] The Bailey rule is a redistribution rule of some payments of Vickery rule maintaining strategy-proofness, decision-efficiency, and individual rationality [Porter et al. (2004) and Cavallo (2006)]. The Bailey rule has excellent features for efficiency and fairness. Not only there exists no Groves rule that Pareto-dominates the Bailey rule [Guo et al. (2013)], but also the Bailey rule satisfies anonymity and other condition of fairness [Porter et al. (2004)]. Although the Bailey rule does not satisfy envy-freeness, this drawback is inevitable among deterministic rules, because any rule satisfying strategy-proofness, anonymity, envy-freeness, and individual rationality is dominated by some strategy-proof rule [Sprumont (2013)].

Other drawback of the Vickery rule is that it does not satisfy *equal rights lower bound*. Equal rights lower bound[8] requires that any agent's assignment be at least better than the equal assignment. This drawback is also not particular to the Vickery rule, because there exists no deterministic rule satisfying strategy-proofness and equal rights lower bound [Moulin (2010)].

Hence, in order to overcome these drawbacks, we need expand the research scope from deterministic rules to probabilistic ones. Among probabilistic rules, there exist many rules satisfying strategy-proofness, budget-balance, equal rights lower bound, and envy-freeness. For example, the rule which always assigns the indivisible good with the equal probability and no monetary transfer to each agent trivially satisfies these desirable properties. However, by Holmström's (1979) theorem, it is impossible to design a probabilistic rule satisfying strategy-proofness and Pareto-efficiency. Thus, the first interesting question we should answer is "what rule satisfying these desirable properties is second best efficient?" After then, the second interesting question is "Is it the only rule satisfying desirable properties?"

To answer the questions, we construct a new rule, called the second price trading rule, which satisfies strategy-proofness, budget-balance, equal rights lower bound, and envy-freeness. Then, we show that this rule is second best efficient. Furthermore, we show that the second price trading rule is only rule satisfying strategy-proofness, equal treatment of equals, *weak decision-efficiency*, *simple generatability*, and either budget-balance or equal rights lower bound. Weak decision-efficiency requires that almost all probability be assigned the agent(s) who has the first highest value, and all probability be assigned the agents who have at least the second highest value. Simple generatability requires that the probability can be generated by a simple

[7]See Porter et al. (2004), Cavallo (2006), Atlamaz and Yengin (2008), Guo and Conitzer (2009), Moulin (2009), Moulin (2010), and Clippel et al. (2014).

[8]This property is called unanimity lower bound by Moulin (2010).

device. We also show that the second price trading rule is only rule satisfying strategy-proofness, equal treatment of equals, weak decision-efficiency, simple generatability, envy-freeness, and *no-trade-no-transfer*. No-trade-no-transfer requires that when all agents get the equal probability, their transfers are zero.

The rest of this paper is organized as follows: Section 2 sets up the model. Section 3 introduces a new rule. Section 4 defines axioms. Section 5 states results. Section 6 verifies independence of axioms. All proofs are provided in Section 7.

2 Model

Let $N = \{1, 2, \ldots, n\}$ be the set of agents, where we assume $n \geq 3$. We consider an environment with a single indivisible good, hereafter called *good*, and one divisible good called *money*. The good can be allocated probabilistically.

Each agent $i \in N$ has a preference over bundles consisting of a probability $s_i \in [0, 1]$ that he gets the good and a monetary transfer $t_i \in \mathbb{R}$ that he receives. We assume that this preference is represented by a utility function $u_i(s_i, t_i) = s_i v_i + t_i$ for some $v_i \in V \equiv \mathbb{R}_+$. Since a preference is identified by v_i, we regard v_i and V as the preference and the set of preferences, respectively. We call a list $v \equiv (v_i)_{i \in N} \in V^n$ a *preference profile*.

The set of *feasible* allocations is

$$Z = \{(s_i, t_i)_{i \in N} \in ([0, 1] \times \mathbb{R})^n : \sum_{i \in N} s_i = 1 \text{ and } \sum_{i \in N} t_i \leq 0\}.$$

A rule is a function $f : V^n \to Z$. Given a rule f and a preference profile $v \in V^n$, we denote by $f_i(v) \equiv (s_i(v), t_i(v)) \in [0, 1] \times \mathbb{R}$ agent i's assignment under $f(v)$. For any $v \in V^n$ and $N' \subseteq N$, let $v_{N'} \in V^{\#N'}$ and $v_{-N'} \in V^{\#N \setminus N'}$ denote $(v_j)_{j \in N'}$ and $(v_j)_{j \notin N'}$, respectively.

3 New Rule

To define a new rule, we need some notation. For any $v \in V^n$, let denote $v_{(1)}$ and $v_{(2)}$ the first and the second highest value among v, respectively. In formally, $v_{(1)} = \max_{i \in N} v_i$ and $v_{(2)} = \max_{i \in N \setminus \{i^*\}} v_i$ where $i^* \in \arg\max_{i \in N} v_i$. So, $v_{(1)} = v_{(2)}$ may occur. For any $v \in V^n$, define $[1_v] = \{i \in N : v_i = v_{(1)}\}$ and $[2_v] = \{i \in N : v_i = v_{(2)}\}$. Notice that $[1_v] = [2_v]$ when $v_{(1)} = v_{(2)}$.

Then, we introduce the new rule. It is just like a trading rule as follow. Initially, all agents have the equal probability $\frac{1}{n}$. They trade the probability at the second price $v_{(2)}$. The agents in $[1_v]$ are buyers, and the others are sellers, except the case $\#[1_v] = 1$ and $\#[2_v] = 1$, where the agent in $[2_v]$ is neither buyer nor seller. Each seller sells the initial probability $\frac{1}{n}$. Each buyer buys equally the sold probability.

Definition **The second price trading rule** is defined as follows. For any $v \in V^n$, when $\#[1_v] = 1$ and $\#[2_v] = 1$,

$$f_i(v) = \begin{cases} (\frac{n-1}{n}, -\frac{n-2}{n} v_{(2)}) & \text{if } i \in [1_v], \\ (\frac{1}{n}, 0) & \text{if } i \in [2_v], \\ (0, \frac{1}{n} v_{(2)}) & \text{otherwise,} \end{cases}$$

and when $\#[1_v] = 1$ and $\#[2_v] > 1$,

$$f_i(v) = \begin{cases} (1, -\frac{n-1}{n} v_{(2)}) & \text{if } i \in [1_v], \\ (0, \frac{1}{n} v_{(2)}) & \text{otherwise,} \end{cases}$$

and when $\#[1_v] > 1$

$$f_i(v) = \begin{cases} (\frac{1}{\#[1_v]}, -(\frac{1}{\#[1_v]} - \frac{1}{n}) v_{(2)}) & \text{if } i \in [1_v], \\ (0, \frac{1}{n} v_{(2)}) & \text{otherwise.} \end{cases}$$

4 Axioms

We introduce some properties that the second price trading rule satisfies. First, *strategy-proofness* says that it is a dominant strategy for any agent to report his true preference.

Definition A rule f satisfies **strategy-proofness (SP)** if for any $v \in V^n$, any $i \in N$, and any $v'_i \in V$, it holds that

$$s_i(v)v_i + t_i(v) \geq s_i(v'_i, v_{-i})v_i + t_i(v'_i, v_{-i}).$$

Second best efficiency says that the rule is in the Pareto frontier among strategy-proof rules.

Definition A strategy-proof rule f is **second best efficient** if there does not exist other strategy-proof rule f^* such that for any $v \in V^n$ and any $i \in N$,

$$s^*_i(v)v_i + t^*_i(v) \geq s_i(v)v_i + t_i(v),$$

and for some $v \in V^n$ and some $j \in N$,

$$s^*_j(v)v_j + t^*_j(v) > s_j(v)v_j + t_j(v).$$

Budget balance says that the transfers among agents are closed.

Definition A rule f satisfies **budget balance (BB)** if for any $v \in V^n$, it holds that

$$\sum_{i \in N} t_i(v) = 0.$$

Equal treatment of equals says that the agents who have the same preference get the same assignment.

Definition A rule f satisfies **equal treatment of equals (ETE)** if for any $v \in V^n$ and any $i, j \in N$, if $v_i = v_j$, then it holds that

$$f_i(v) = f_j(v).$$

Equal rights lower bound says that the assignment is at least better than the equal assignment $(\frac{1}{n}, 0)$.

Definition A rule f satisfies **equal rights lower bound (ERLB)** if for any $v \in V^n$ and any $i \in N$, it holds that

$$s_i(v)v_i + t_i(v) \geq \frac{1}{n}v_i.$$

Envy-freeness says that no agent prefers another agent's assignment to his own assignment.

Definition A rule f satisfies **envy-freeness (EF)** if for any $v \in V^n$ and any $i, j \in N$, it holds that

$$s_i(v)v_i + t_i(v) \geq s_j(v)v_i + t_j(v).$$

No-trade-no-transfer says that when all agents get the equal probability $\frac{1}{n}$, their transfers are zero.

Definition A rule f satisfies **no-trade-no-transfer (NTNT)** if for any $v \in V^n$ if for any $i \in N$,

$$s_i(v) = \frac{1}{n},$$

then for any $i \in N$, it holds that

$$t_i(v) = 0.$$

Weak decision-efficiency says that almost all probability is assigned the agent(s) whose valuation is the first highest, and all probability is assigned the agents whose valuations are at least the second highest.

Definition A rule f satisfies **weak decision-efficiency (wDE)** if for any $v \in V^n$, it holds that

$$\sum_{i \in [1]} s_i(v) \geq \frac{n-1}{n},$$

and

$$\sum_{i \in [1] \cup [2]} s_i(v) = 1.$$

Simple generatability says that the probability can be generated by a simple device, like n balls.

Definition A rule f satisfies **simple generatability (SG)** if for any $v \in V^n$ and any $i \in N$, there exist some non-negative integers $m, m' \leq n$ such that

$$s_i(v) = \frac{m'}{m}.$$

5 Results

We state the results. All proofs are provided in the final section. The first result says that the new rule satisfies our main axiom, strategy-proofness.

Theorem 1 The second price trading rule satisfies strategy-proofness.

The next result says that the new rule is in the frontier among strategy-proof rules.

Theorem 2 The second price trading rule is second best efficient.

From the above result, we can say that the new rule is not bad. To say that the new rule is good, we need to show that the new rule has a special feature. The next three results say that the new rule is the only rule satisfying good properties.

Theorem 3 A rule satisfies strategy-proofness, budget-balance, equal treatment of equals, weak decision-efficiency, and simple generatability if and only if it is the second price trading rule.

Theorem 4 A rule satisfies strategy-proofness, equal rights lower bound, equal treatment of equals, weak decision-efficiency, and simple generatability if and only if it is the second price trading rule.

Theorem 5 A rule satisfies strategy-proofness, envy-freeness, no-trade-no-transfer, equal treatment of equals, weak decision-efficiency, and simple generatability if and only if it is the second price trading rule.

6 Independence of Axioms

We verify that none of the axioms in Theorems 3, 4, and 5 is redundant. We exhibit rules that satisfy all but one of the axioms. Let $n = 3$.

Example 1 (not SP) Let f be as follows: for any $v \in V^3$ and any $i \in N$, when $\#[1_v] = 3$,

$$f_i(v) = (\frac{1}{3}, 0),$$

and when $\#[1_v] < 3$,

$$f_i(v) = \begin{cases} (\frac{1}{\#[1_v]}, -(\frac{1}{\#[1_v]} - \frac{1}{3})v_{(1)}) & \text{if } i \in [1_v], \\ (0, \frac{1}{3}v_{(1)}) & \text{otherwise.} \end{cases}$$

This rule satisfies all but not strategy-proofness.

Example 2 (not ETE) Let f be as follows: for any $v \in V^3$, when $v = (0, 0, 0)$,

$$f_1(v) = (1, 0) \text{ and } f_2(v) = f_3(v) = (0, 0),$$

and when $v \neq (0, 0, 0)$,

$$f(v) \text{ is determined by the second price trading rule.}$$

This rule satisfies all but not equal treatment of equals.

Example 3 (not wDE) Let f be as follows: for any $v \in V^3$ and any $i \in N$,

$$f_i(v) = (\frac{1}{3}, 0),$$

This rule satisfies all but not weak decision-efficiency.

Example 4 (not SG) Let f be as follows: for any $v \in V^3$ and any $i \in N$, when for some $\alpha > 0$, $v = (\alpha, 0, 0)$, $v = (0, \alpha, 0)$, or $v = (0, 0, \alpha)$,

$$f_i(v) = \begin{cases} (\frac{2}{3}, 0) & \text{if } i \in [1_v], \\ (\frac{1}{6}, 0) & \text{otherwise,} \end{cases}$$

and when the other cases,

$$f(v) \text{ is determined by the second price trading rule.}$$

This rule satisfies all but not simple generatability.

Example 5 (not BB, not ERLB, not NTNT) Given $\alpha > 0$. Let f be as follows: for any $v \in V^3$ and any $i \in N$,

$$s_i(v) \text{ is determined by the second price trading rule,}$$

and

$$t_i(v) = -\alpha + \text{ his transfer determined by the second price trading rule.}$$

This rule satisfies all but not budget-balance, not equal rights lower bound, and not no-trade-no-transfer.

Example 6 (not BB, not ERLB, not EF) Given $\alpha > 0$. Let f be as follows: for any $v \in V^3$ and any $i \in N$,

$$s_i(v) \text{ is determined by the second price trading rule,}$$

and when $\#\{j \in N : v_j = 0\} = 2$ and $i \in \{j \in N : v_j = 0\}$,

$$t_i(v) = -\alpha + \text{ his transfer determined by the second price trading rule,}$$

and when $\#\{j \in N : v_j = 0\} = 2$ and $i \notin \{j \in N : v_j = 0\}$,

$$t_i(v) = \text{ his transfer determined by the second price trading rule,}$$

and when $\#\{j \in N : v_j = 0\} = 1$ and $i \in \{j \in N : v_j = 0\}$,

$$t_i(v) = \text{ his transfer determined by the second price trading rule,}$$

and when $\#\{j \in N : v_j = 0\} = 1$ and $i \notin \{j \in N : v_j = 0\}$,

$$t_i(v) = -\alpha + \text{ his transfer determined by the second price trading rule,}$$

and when the other cases,

$$t_i(v) = \text{ his transfer determined by the second price trading rule.}$$

This rule satisfies all but not budget-balance, not equal rights lower bound, and not envy-freeness.

7 Proofs

Throughout the all proofs, we use the following Lemma which have been shown by Myerson (1981).

Lemma (Myerson, 1981) A rule f satisfies strategy-proofness if and only if for any $i \in N$, any $v_i, v_i' \in V$ such that $v_i \leq v_i'$, and any $v_{-i} \in V^{n-1}$, it holds that

$$s_i(v_i, v_{-i}) \leq s_i(v_i', v_{-i}),$$

and that

$$t_i(v_i, v_{-i}) = t_i(0, v_{-i}) - s_i(v_i, v_{-i})v_i + \int_0^{v_i} s_i(x_i, v_{-i})dx_i.$$

From this, we also have the following easily.[9] If a rule f satisfies strategy-proofness, then for any $i \in N$, any $v_i, v_i' \in V$ such that $v_i \leq v_i'$, and any $v_{-i} \in V^{n-1}$, it holds that

$$t_i(v_i', v_{-i}) = t_i(v_i, v_{-i}) - s_i(v_i', v_{-i})v_i' + s_i(v_i, v_{-i})v_i + \int_{v_i}^{v_i'} s_i(x_i, v_{-i})dx_i.$$

7.1 Proof of Theorem 1

We show that the second price trading rule satisfies *strategy-proofness*. Let f denote the second price trading rule. Let $i \in N$. Let $v \in V^n$. We divide the argument into two cases.

Case 1. *The number of agent whose valuation is the first highest in* $N \setminus \{i\}$ *is* 1.
Let $v_{i_1}, v_{i_2} \in V$ denote the first and the second highest valuation in $N \setminus \{i\}$, respectively. Note that

$$s_i(v) = \begin{cases} 0 & \text{if } v_i \leq v_{i_2}, \\ \frac{1}{n} & \text{if } v_{i_2} < v_i < v_{i_1}, \\ \frac{1}{2} & \text{if } v_i = v_{i_1}, \\ \frac{n-1}{n} & \text{if } v_{i_1} < v_i. \end{cases}$$

Note also that

$$t_i(0, v_{-i}) = \frac{1}{n} v_{i_2}.$$

[9] Subtract $t_i(v_i, v_{-i})$ from $t_i(v_i', v_{-i})$.

Then, it follows that

$$t_i(0,v_{-i})-s_i(v_i,v_{-i})v_i+\int_0^{v_i} s_i(x_i,v_{-i})dx_i = \begin{cases} \frac{1}{n}v_{i_2} & \text{if } v_i \leq v_{i_2}, \\ 0 & \text{if } v_{i_2} < v_i < v_{i_1}, \\ -(\frac{1}{2}-\frac{1}{n})v_{i_1} & \text{if } v_i = v_{i_1}, \\ -\frac{n-2}{n}v_{i_1} & \text{if } v_{i_1} < v_i. \end{cases}$$

Thus, we have $t_i(v_i,v_{-i}) = t_i(0,v_{-i}) - s_i(v_i,v_{-i})v_i + \int_0^{v_i} s_i(x_i,v_{-i})dx_i$. Then, Myerson's Lemma implies that f satisfies *strategy-proofness.*

Case 2. *The number of agents whose valuations are the first highest in* $N \setminus \{i\}$ *is more than* 1.

Let m denote the number of agents whose valuations are the first highest in $N \setminus \{i\}$. Let $v_{i_1} \in V$ denote the first highest valuation in $N \setminus \{i\}$. Note that

$$s_i(v) = \begin{cases} 0 & \text{if } v_i < v_{i_1}, \\ \frac{1}{m+1} & \text{if } v_i = v_{i_1}, \\ 1 & \text{if } v_{i_1} < v_i. \end{cases}$$

Note also that

$$t_i(0,v_{-i}) = \frac{1}{n}v_{i_1}.$$

Then, it follows that

$$t_i(0,v_{-i}) - s_i(v_i,v_{-i})v_i + \int_0^{v_i} s_i(x_i,v_{-i})dx_i = \begin{cases} \frac{1}{n}v_{i_1} & \text{if } v_i < v_{i_1}, \\ -(\frac{1}{m+1}-\frac{1}{n})v_{i_1} & \text{if } v_i = v_{i_1}, \\ -\frac{n-1}{n}v_{i_1} & \text{if } v_{i_1} < v_i. \end{cases}$$

Thus, we have $t_i(v_i,v_{-i}) = t_i(0,v_{-i}) - s_i(v_i,v_{-i})v_i + \int_0^{v_i} s_i(x_i,v_{-i})dx_i$. Then, Myerson's Lemma implies that f satisfies *strategy-proofness.*

□

7.2 Proof of Theorem 2

We show that the second price trading rule is the second best efficient. Let f denote the second price trading rule. Let f^* be a strategy-proof rule as follows: for any $v \in V^n$, it holds that

$$s_i^*(v)v_i + t_i^*(v) \geq s_i(v)v_i + t_i(v). \tag{1}$$

Let $v \in V^n$. For simplicity of notation, we assume $1 \in [1_v]$ and $2 \in [2_v]$. We divide the argument into three cases.

Case 1: $\#[1_v] = 1$ *and* $\#[2_v] > 1$.
Note that

$$f_i(v) = \begin{cases} (1, -\frac{n-1}{n}v_2) & \text{if } i \in [1_v], \\ (0, \frac{1}{n}v_2) & \text{otherwise.} \end{cases}$$

We claim that $s_1^*(v) = 1$. Suppose to the contrary that $s_1^*(v) < 1$. Then, from (1), we have

$$s_1^*(v)v_2 + t_1^*(v) > s_1(v)v_2 + t_1(v).$$

For any $i \neq 1$, from (1), we also have

$$s_i^*(v)v_2 + t_i^*(v) \geq s_i(v)v_2 + t_i(v).$$

By summing up these inequalities, it follows that

$$v_2 > v_2,$$

which is a contradiction. So, we have

$$s_1^*(v) = 1.$$

This implies for any $i \neq 1$,

$$s_i^*(v) = 0.$$

So, for any $i \in N$, it holds that

$$t_i^*(v) = t_i(v).$$

Case 2: $\#[1_v] = 1$ *and* $\#[2_v] = 1$.
Note that

$$f_i(v) = \begin{cases} (\frac{n-1}{n}, -\frac{n-2}{n}v_2) & \text{if } i \in [1_v], \\ (\frac{1}{n}, 0) & \text{if } i \in [2_v], \\ (0, \frac{1}{n}v_2) & \text{otherwise,} \end{cases}$$

By case 1, it holds that

$$f_2^*(\hat{v}_2, v_{-2}) = (0, \frac{1}{n}\hat{v}_2),$$

where $\hat{v}_2 = v_{(3)}$. Since f^* satisfies *SP*, it follows that

$$\frac{1}{n}\hat{v}_2 \geq s_2^*(v)\hat{v}_2 + t_2^*(v). \tag{2}$$

By combining the inequalities (1) and (2), we have

$$(s_2^*(v) - \frac{1}{n})(v_2 - \hat{v}_2) \geq 0.$$

Since $v_2 - \hat{v}_2 > 0$, this implies that

$$s_2^*(v) \geq \frac{1}{n}.$$

We claim that $s_2^*(v) = \frac{1}{n}$. Suppose to the contrary that $s_2^*(v) > \frac{1}{n}$. Then, for any $i \in N$, from (1), we have

$$s_i^*(v)v_1 + t_i^*(v) \geq s_i(v)v_1 + t_i(v),$$

where the inequality is strict for agent 2. By summing up these inequalities, it follows that

$$v_1 > v_1,$$

which is a contradiction. So, we have

$$s_2^*(v) = \frac{1}{n}.$$

This implies that

$$s_1^*(v) \leq \frac{n-1}{n}.$$

We claim that $s_1^*(v) = \frac{n-1}{n}$. Suppose to the contrary that $s_1^*(v) < \frac{n-1}{n}$. Then, from (1), we have

$$s_1^*(v)v_2 + t_1^*(v) > s_1(v)v_2 + t_1(v).$$

For any $i \neq 1$, from (1), we also have

$$s_i^*(v)v_2 + t_i^*(v) \geq s_i(v)v_2 + t_i(v).$$

By summing up these inequalities, it follows that

$$v_2 > v_2,$$

which is a contradiction. So, we have

$$s_1^*(v) = \frac{n-1}{n}.$$

This implies for any $i \neq 1, 2$,

$$s_i^*(v) = 0.$$

So, for any $i \in N$, it holds that

$$t_i^*(v) = t_i(v).$$

Case 3: $\#[1_v] > 1$.
Note that

$$f_i(v) = \begin{cases} (\frac{1}{\#[1_v]}, -(\frac{1}{\#[1_v]} - \frac{1}{n})v_1) & \text{if } i \in [1_v], \\ (0, \frac{1}{n}v_1) & \text{otherwise,} \end{cases}$$

We claim that for any $j \notin [1_v]$, $s_j^*(v) = 0$. Suppose to the contrary that for some $h \notin [1_v]$, $s_h^*(v) > 0$. Then, from (1), we have

$$s_h^*(v)v_1 + t_h^*(v) > s_h(v)v_1 + t_h(v).$$

For any $j \notin [1_v]$, from (1), we also have

$$s_j^*(v)v_1 + t_j^*(v) \geq s_j(v)v_1 + t_j(v).$$

Note that

$$\sum_{i \in [1_v]} s_i^*(v)v_1 + \sum_{i \in [1_v]} t_i^*(v) \geq \sum_{i \in [1_v]} s_i(v)v_1 + \sum_{i \in [1_v]} t_i(v).$$

By summing up these inequalities, we have

$$\sum_{i \in N} s_i^*(v)v_1 + \sum_{i \in N} t_i^*(v) > \sum_{i \in N} s_i(v)v_1 + \sum_{i \in N} t_i(v),$$

which implies $v_1 > v_1$, a contradiction. So, for any $j \notin [1_v]$, we have

$$s_j^*(v) = 0.$$

Then, for any $j \notin [1_v]$, it follows that

$$t_j^*(v) = t_j(v).$$

These imply that for any $i \in [1_v]$, it holds that

$$s_i^*(v)v_i + t_i^*(v) = s_i(v)v_i + t_i(v).$$

Thus, f is the second best efficient.

□

7.3 Proof of Theorem 3

In the following, for any partition (I, C, O) of N where some set may be empty, we use the notation $v = (v^1_I, v_C, v^0_O)$ in which for any $i, j \in I$, any $k \in C$, and any $h, h' \in O$, $v^1_i = v^1_j > v_k > v^0_h = v^0_{h'}$, where v^1_i and v^0_h are any values in V.

Let f be a rule satisfying *SP*, *BB*, *ETE*, *wDE*, and *SG*. We show that for any $v \in V^n$, $f(v)$ coincides with the allocation determined by the second price trading rule. To do so, we prove the following induction.

1. (A^0) For any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C = 1$ and $\#I = 1$, $f(v^1_I, v_C, v^0_O)$ coincides with the allocation determined by the second price trading rule.
   (B^0) For any $(v^1_I, v^0_O) \in V^n$, $f(v^1_I, v^0_O)$ also do.

2. Given any integer c such that $2 \leq c \leq n-2$. If
   (A) for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C \leq c-1$ and $\#I = 1$, $f(v^1_I, v_C, v^0_O)$ coincides with the allocation determined by the second price trading rule, and
   (B) for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C \leq c-2$ and $I \neq \emptyset$, $f(v^1_I, v_C, v^0_O)$ also do, then
   (A') for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C = c$ and $\#I = 1$, $f(v^1_I, v_C, v^0_O)$ also do, and
   (B') for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C = c-1$ and $I \neq \emptyset$, $f(v^1_I, v_C, v^0_O)$ also do.

The First Part.

Before proving (A^0) and (B^0), we show preliminary results. Pick up any two agents, say $1, 2 \in N$, and set $O = N \setminus \{1, 2\}$. Let $v^1_i, v^0_i \in V$ be such that $v^1_i > v^0_i$. By *ETE* and *BB*, we have for any $i \in N$,

$$f_i(v^0_{O\cup\{1,2\}}) = (\frac{1}{n}, 0).$$

By *wDE*, *SG*, and *ETE*, we have

$$s_1(v^1_1, v^0_{O\cup\{2\}}) = 1.$$

By Myerson's Lemma, it holds that

$$t_1(v^1_1, v^0_{O\cup\{2\}}) = -\frac{n-1}{n} v^0_2.$$

Then, by *ETE* and *BB*, it follows that for any $i \neq 1$,

$$f_i(v^1_1, v^0_{O\cup\{2\}}) = (0, \frac{1}{n} v^0_2).$$

So, $f(v_1^1, v^0_{O\cup\{2\}})$ coincides with the allocation determined by the second price trading rule. By *wDE* and *ETE*, we have

$$s_1(v^1_{\{1,2\}}, v^0_O) = s_2(v^1_{\{1,2\}}, v^0_O) = \frac{1}{2}.$$

By *wDE* and *SG*, for any $\hat{v}_2 \in V$ such that $v_i^0 < \hat{v}_2 < v_1^1$, it follows that $s_2(v_1^1, \hat{v}_2, v_O^0)$ is either 0 or $\frac{1}{n}$.

We claim that for any such $\hat{v}_2$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_O^0) = \frac{1}{n}.$$

Suppose to the contrary that for some $\hat{v}_2 \in V$ such that $v_i^0 < \hat{v}_2 < v_1^1$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_O^0) = 0.$$

Then, by Myerson's Lemma, there exists $a \in V$ such that $\hat{v}_2 \le a \le v_1^1$, and the following is satisfied: For any $v_2' \in V$ such that $v_i^0 < v_2' < a$, we have

$$s_2(v_1^1, v_2', v_O^0) = 0$$

and for any $v_2' \in V$ such that $a < v_2' < v_1^1$, we have

$$s_2(v_1^1, v_2', v_O^0) = \frac{1}{n}.$$

Furthermore, Myerson's Lemma gives that

$$t_2(v^1_{\{1,2\}}, v^0_O) \neq -(\frac{1}{2} - \frac{1}{n})v_1^1.$$

By *ETE* and *BB*, it holds that for any $i \neq 1, 2$,

$$t_i(v^1_{\{1,2\}}, v^0_O) \neq \frac{1}{n}v_1^1.$$

By *wDE* and *ETE*, we have

$$s_3(v^1_{\{1,2,3\}}, v^0_{O\setminus\{3\}}) = \frac{1}{3}.$$

By *wDE*, for any $\hat{v}_3 \in V$ such that $v_i^0 \le \hat{v}_3 < v_1^1$, it follows that

$$s_3(v^1_{\{1,2\}}, \hat{v}_3, v^0_{O\setminus\{3\}}) = 0.$$

Then, Myerson's Lemma gives that

$$t_3(v^1_{\{1,2,3\}}, v^0_{O\setminus\{3\}}) \neq -(\frac{1}{3} - \frac{1}{n})v_1^1.$$

Repeating the same argument, we have

$$t_n(v_N^1) \neq -(\frac{1}{n} - \frac{1}{n})v_1^1 = 0.$$

Since, by *ETE* and *BB*, it must be $t_n(v_N^1) = 0$, this is contradiction. Thus, for any $\hat{v}_2 \in V$ such that $v_i^0 < \hat{v}_2 < v_1^1$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_O^0) = \frac{1}{n}.$$

Then, by Myerson's Lemma, we have

$$f_2(v_1^1, v_2, v_O^0) = \begin{cases} (\frac{1}{2}, -(\frac{1}{2} + \frac{1}{n})v_1^1) & \text{if } v_2 = v_1^1, \\ (\frac{1}{n}, 0) & \text{if } v_1^1 > v_2 > v_i^0. \end{cases} \quad (3)$$

The (A^0) ***Part***.

Let $(v_I^1, v_C, v_O^0) \in V^n$ be such that $\#C = 1$ and $\#I = 1$. We denote $I = \{i_1\}$ and $C = \{i_2\}$. From (3), we have $f_{i_2}(v_I^1, v_C, v_O^0) = (\frac{1}{n}, 0)$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_O^0) = \frac{n-1}{n}.$$

Since, from (3), $f_{i_1}(\hat{v}_{i_1}, v_C, v_O^0) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_O^0) = -\frac{n-2}{n}v_{i_2}.$$

Then, by *BB* and *ETE*, for any $h \in O$, it holds that

$$f_h(v_I^1, v_C, v_O^0) = (0, \frac{1}{n}v_{i_2}).$$

Thus, (A^0) is valid.

The (B^0) ***Part***.

Let $v = (v_I^1, v_O^0) \in V^n$. When $\#I = 0$ or 1, we have already shown as the preliminary results. So, consider the case of $\#I > 1$.

Let $i, j \in I$. From (3), it follows that

$$f_i(v_{\{i,j\}}^1, v_{O \cup I \setminus \{i,j\}}^0) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_i^1).$$

Then, by *BB* and *ETE*, for any $h \in O \cup I \setminus \{i, j\}$, it follows that

$$f_h(v_{\{i,j\}}^1, v_{O \cup I \setminus \{i,j\}}^0) = (0, \frac{1}{n}v_i^1),$$

that is, $f(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}})$ coincides with the allocation determined by the second price trading rule. By the similar way, we can show that for any $k \in I \setminus \{i,j\}$, $f(v^1_{\{i,j,k\}}, v^0_{O\cup I\setminus\{i,j,k\}})$ coincides with the allocation determined by the second price trading rule. Repeating the same argument, we have (B^0). Thus, the first part is valid.

The Second Part.
Given any integer c such that $2 \leq c \leq n-2$. Before proving (A') and (B'), we show preliminary results. Let $(v^1_I, v_{C'}, v^0_O) \in V^n$ be such that $\#C' = c-1$ and $\#I = 1$. For simplicity of notation, we denote $I = \{1\}$, and v_2 as the highest valuation in C'. Pick up any agent $h \in O$.

Note that by *wDE*, *SG*, and *ETE*, we have

$$s_h(v_h, v^1_I, v_{C'}, v^0_{O\setminus\{h\}}) = \begin{cases} 0 & \text{if } v^0_h \leq v_h \leq v_2, \\ 0 \text{ or } \frac{1}{n} & \text{if } v_2 < v_h < v^1_1, \\ \frac{1}{2} & \text{if } v_h = v^1_1. \end{cases}$$

We claim that for any $v_h \in V$ such that $v_2 < v_h < v^1_1$, we have

$$s_h(v_h, v^1_I, v_{C'}, v^0_{O\setminus\{h\}}) = \frac{1}{n}.$$

Suppose to the contrary that for some $\hat{v}_h \in V$ such that $v_2 < \hat{v}_h < v^1_1$, it holds that

$$s_h(\hat{v}_h, v^1_I, v_{C'}, v^0_{O\setminus\{h\}}) = 0.$$

Then, by Myerson's Lemma, there exists $a \in V$ such that $\hat{v}_h \leq a \leq v^1_1$, and the following is satisfied: For any $v'_h \in V$ such that $v^0_h < v'_h < a$, we have

$$s_h(v'_h, v^1_I, v_{C'}, v^0_{O\setminus\{h\}}) = 0$$

and for any $v'_h \in V$ such that $a < v'_h < v^1_1$, we have

$$s_h(v'_h, v^1_I, v_{C'}, v^0_{O\setminus\{h\}}) = \frac{1}{n}.$$

Since, by the induction hypothesis (A), $f_h(v^1_I, v_{C'}, v^0_O) = (0, \frac{1}{n}v_2)$, Myerson's Lemma also gives that

$$t_h(v^1_{I\cup\{h\}}, v_{C'}, v^0_{O\setminus\{h\}}) \neq -(\frac{1}{2} - \frac{1}{n})v^1_1.$$

Note that, by *wDE*, for any $j \in C'$, $s_j(v^1_{I\cup\{h\}}, v_{C'}, v^0_{O\setminus\{h\}}) = 0$. Since, by the induction hypothesis (B), for any $j \in C'$, $f_j(v^1_{I\cup\{h\}}, v_{C'\setminus\{j\}}, v^0_{O\cup\{j\}\setminus\{h\}}) = (0, \frac{1}{n}v^1_1)$, by strategy-proofness, it holds that

$$t_j(v^1_{I\cup\{h\}}, v_{C'}, v^0_{O\setminus\{h\}}) = \frac{1}{n}v^1_1.$$

By *ETE* and *BB*, it holds that for any $i \in O \setminus \{h\}$,

$$t_i(v^1_{I \cup \{h\}}, v_{C'}, v^0_{O \setminus \{h\}}) \neq \frac{1}{n} v^1_1.$$

Pick up any agent $h' \in O \setminus \{h\}$. By *wDE*, *SG*, and *ETE*, we have

$$s_{h'}(v^1_{I \cup \{h,h'\}}, v_{C'}, v^0_{O \setminus \{h,h'\}}) = \frac{1}{3}.$$

By *wDE*, for any $\hat{v}_{h'} \in V$ such that $v^0_{h'} \leq \hat{v}_{h'} < v^1_1$, it follows that

$$s_{h'}(\hat{v}_{h'}, v^1_{I \cup \{h\}}, v_{C'}, v^0_{O \setminus \{h,h'\}}) = 0.$$

Then, Myerson's Lemma gives that

$$t_{h'}(v^1_{I \cup \{h,h'\}}, v_{C'}, v^0_{O \setminus \{h,h'\}}) \neq -(\frac{1}{3} - \frac{1}{n}) v^1_1.$$

Repeating the same argument, we have for any $i \in I \cup O$,

$$s_i(v^1_{I \cup O}, v_{C'}) = \frac{1}{n - (c-1)}$$

and

$$t_i(v^1_{I \cup O}, v_{C'}) \neq -(\frac{1}{n-(c-1)} - \frac{1}{n}) v^1_1.$$

By the induction hypothesis (B) and *SP*, for any $k \in C'$, it holds that

$$t_k(v^1_{I \cup O}, v_{C'}) = \frac{1}{n} v^1_1.$$

These, however, contradict *BB*. Thus, for any $v_h \in V$ such that $v_2 < v_h < v^1_1$, we have

$$s_h(v_h, v^1_I, v_{C'}, v^0_{O \setminus \{h\}}) = \frac{1}{n}.$$

Then, by Myerson's Lemma, we have

$$f_h(v_h, v^1_I, v_{C'}, v^0_{O \setminus \{h\}}) = \begin{cases} (0, \frac{1}{n} v_2) & \text{if } v^0_h \leq v_h \leq v_2 \\ (\frac{1}{n}, 0) & \text{if } v_2 < v_h < v^1_1 \\ (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n}) v^1_1) & \text{if } v_h = v^1_1. \end{cases} \quad (4)$$

The (A') ***Part***.
Let $v = (v^1_I, v_C, v^0_O) \in V^n$ be such that $\#C = c$ and $\#I = 1$. We denote $I = \{i_1\}$, and v_{i_2} as the highest valuation in C, that is, $i_2 \in [2_v]$. We divide the argument into two cases.

The case 1: $\#[2_v] = 1$.
From (4), we have $f_{i_2}(v_I^1, v_C, v_O^0) = (\frac{1}{n}, 0)$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_O^0) = \frac{n-1}{n}.$$

Since, from (4), $f_{i_1}(\hat{v}_{i_1}, v_C, v_O^0) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_O^0) = -\frac{n-2}{n}v_{i_2}.$$

Then, for any $i \neq i_1, i_2$, it holds that

$$s_i(v_I^1, v_C, v_O^0) = 0.$$

By the induction hypothesis (A), for any $k \in C \setminus \{i_2\}$, it follows that

$$f_k(v_I^1, v_{C\setminus\{k\}}, v_{O\cup\{k\}}^0) = (0, \frac{1}{n}v_{i_2}).$$

So, by *SP*, for any $k \in C \setminus \{i_2\}$, it holds that

$$t_k(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

Then, by *BB*, for any $h \in O$, it follows that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

The case 2: $\#[2_v] > 1$.
From (4), for any $i \in [2_v]$, we have $f_i(v_I^1, v_C, v_O^0) = (0, \frac{1}{n}v_{i_2})$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_O^0) = 1.$$

Since, by the induction hypothesis (B), $f_{i_1}(\hat{v}_{i_1}, v_C, v_O^0) = (\frac{1}{\#[2_v]+1}, -(\frac{1}{\#[2_v]+1} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_O^0) = -\frac{n-1}{n}v_{i_2}.$$

Then, for any $i \notin [1_v] \cup [2_v]$, it holds that

$$s_i(v_I^1, v_C, v_O^0) = 0.$$

By the induction hypothesis (A), for any $k \in C \setminus [2_v]$, it follows that

$$f_k(v_I^1, v_{C\setminus\{k\}}, v_{O\cup\{k\}}^0) = (0, \frac{1}{n}v_{i_2}).$$

So, by *SP*, for any $k \in C \setminus [2_v]$, it holds that

$$t_i(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

Then, by *BB* and *ETE*, for any $h \in O$, it also follows that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

Thus, (A') is valid.

The (B') ***Part***.

Let $v = (v_I^1, v_C, v_O^0) \in V^n$ be such that $\#C = c - 1$ and $I \neq \emptyset$. If $\#I = 1$, then the induction hypothesis (A) implies the conclusion. So, consider the case of $\#I > 1$.

Let $i, j \in I$. Let $i_2 \in C$ be such that his valuation v_{i_2} is the highest in C. From (A'), it holds that

$$f_j(\hat{v}_j, v_i^1, v_C, v_{I\setminus\{i,j\}}^0, v_O^0) = (0, \frac{1}{n}v_{i_2})$$

where $\hat{v}_j = v_{i_2}$, and that for any $v_j' \in V$ such that $v_i^1 > v_j' > v_{i_2}$,

$$s_j(v_j', v_i^1, v_C, v_{I\setminus\{i,j\}}^0, v_O^0) = \frac{1}{n}.$$

Since, by *wDE* and *ETE*, $s_j(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^0, v_O^0) = \frac{1}{2}$, by Myerson's Lemma, it follows that

$$t_j(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^0, v_O^0) = -(\frac{1}{2} - \frac{1}{n})v_i^1.$$

Then, for any $k \neq i, j$, we have

$$s_k(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^0, v_O^0) = 0.$$

Note that by the induction hypothesis (B), for any $k \in C$, it follows that

$$f_k(v_{\{i,j\}}^1, v_{C\setminus\{k\}}, v_{I\setminus\{i,j\}}^0, v_{O\cup\{k\}}^0) = (0, \frac{1}{n}v_i^1).$$

So, by *SP*, for any $k \in C$, we have

$$t_k(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^0, v_O^0) = \frac{1}{n}v_i^1.$$

Then, by *BB* and *ETE*, for any $h \in O \cup I \setminus \{i,j\}$, it also follows that

$$t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) = \frac{1}{n}v^1_i,$$

that is, $f(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O)$ coincides with the allocation determined by the second price trading rule. By the similar way, we can show that for any $k \in I \setminus \{i,j\}$, $f(v^1_{\{i,j,k\}}, v_C, v^0_{I\setminus\{i,j,k\}}, v^0_O)$ coincides with the allocation determined by the second price trading rule. Repeating the same argument, we have (B'). Thus, the second part is valid. Therefore, this theorem is valid.

□

7.4 Proof of Theorem 4

Let f be a rule satisfying *SP*, *ERLB*, *ETE*, *wDE*, and *SG*. We show that for any $v \in V^n$, $f(v)$ coincides with the allocation determined by the second price trading rule. To do so, we prove the following induction.

1. (A^0) For any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C = 1$ and $\#I = 1$, $f(v^1_I, v_C, v^0_O)$ coincides with the allocation determined by the second price trading rule.
   (B^0) For any $(v^1_I, v^0_O) \in V^n$, $f(v^1_I, v^0_O)$ also do.

2. Given any integer c such that $2 \le c \le n-2$. If
   (A) for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C \le c-1$ and $\#I = 1$, $f(v^1_I, v_C, v^0_O)$ coincides with the allocation determined by the second price trading rule, and
   (B) for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C \le c-2$ and $I \ne \emptyset$, $f(v^1_I, v_C, v^0_O)$ also do, then
   (A') for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C = c$ and $\#I = 1$, $f(v^1_I, v_C, v^0_O)$ also do, and
   (B') for any $(v^1_I, v_C, v^0_O) \in V^n$ such that $\#C = c-1$ and $I \ne \emptyset$, $f(v^1_I, v_C, v^0_O)$ also do.

The First Part.
Before proving (A^0) and (B^0), we show preliminary results. Pick up any two agents, say $1, 2 \in N$, and set $O = N \setminus \{1,2\}$. Let $v^1_i, v^0_i \in V$ be such that $v^1_i > v^0_i$. By *ETE* and *ERLB*, we have for any $i \in N$,

$$f_i(v^0_{O\cup\{1,2\}}) = (\frac{1}{n}, 0).$$

By *wDE*, *SG*, and *ETE*, we have

$$s_1(v_1^1, v_{O\cup\{2\}}^0) = 1.$$

By Myerson's Lemma, it holds that

$$t_1(v_1^1, v_{O\cup\{2\}}^0) = -\frac{n-1}{n}v_2^0.$$

Then, by *ETE*, *ERLB*, and feasibility of transfer, it follows that for any $i \neq 1$,

$$f_i(v_1^1, v_{O\cup\{2\}}^0) = (0, \frac{1}{n}v_2^0).$$

So, $f(v_1^1, v_{O\cup\{2\}}^0)$ coincides with the allocation determined by the second price trading rules. By *wDE* and *ETE*, we have

$$s_1(v_{\{1,2\}}^1, v_O^0) = s_2(v_{\{1,2\}}^1, v_O^0) = \frac{1}{2}.$$

By *wDE* and *SG*, for any $\hat{v}_2 \in V$ such that $v_i^0 < \hat{v}_2 < v_1^1$, it follows that $s_2(v_1^1, \hat{v}_2, v_O^0)$ is either 0 or $\frac{1}{n}$.

We claim that for any such $\hat{v}_2$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_O^0) = \frac{1}{n}.$$

Suppose to the contrary that for some $\hat{v}_2 \in V$ such that $v_i^0 < \hat{v}_2 < v_1^1$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_O^0) = 0.$$

Then, by *SP*, we have

$$t_2(v_1^1, \hat{v}_2, v_O^0) = \frac{1}{n}v_2^0,$$

which contradicts *ERLB*. Thus, for any $\hat{v}_2 \in V$ such that $v_i^0 < \hat{v}_2 < v_1^1$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_O^0) = \frac{1}{n}.$$

Then, by Myerson's Lemma, we have

$$f_2(v_1^1, v_2, v_O^0) = \begin{cases} (\frac{1}{2}, -(\frac{1}{2} + \frac{1}{n})v_1^1) & \text{if } v_2 = v_1^1, \\ (\frac{1}{n}, 0) & \text{if } v_1^1 > v_2 > v_i^0. \end{cases} \tag{5}$$

The (A^0) ***Part***.

Let $(v_I^1, v_C, v_O^0) \in V^n$ be such that $\#C = 1$ and $\#I = 1$. We denote $I = \{i_1\}$

and $C = \{i_2\}$. From (5), we have $f_{i_2}(v_I^1, v_C, v_O^0) = (\frac{1}{n}, 0)$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_O^0) = \frac{n-1}{n}.$$

Since, from (5), $f_{i_1}(\hat{v}_{i_1}, v_C, v_O^0) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_O^0) = -\frac{n-2}{n}v_{i_2}.$$

Then, for any $h \in O$, it holds that

$$s_h(v_I^1, v_C, v_O^0) = 0.$$

By *ETE* and feasibility of transfer, for any $h \in O$, it also holds that

$$t_h(v_I^1, v_C, v_O^0) \leq \frac{1}{n}v_{i_2}.$$

We claim that $t_h(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}$. Suppose to the contrary that

$$t_h(v_I^1, v_C, v_O^0) < \frac{1}{n}v_{i_2}.$$

By *wDE*, *SG*, and *ETE*, we have

$$s_h(\hat{v}_h, v_I^1, v_C, v_{O\setminus\{h\}}^0) = 0,$$

where $\hat{v}_h = v_{i_2}$. Then, by *SP*, it holds that

$$t_h(\hat{v}_h, v_I^1, v_C, v_{O\setminus\{h\}}^0) = t_h(v_I^1, v_C, v_O^0) < \frac{1}{n}v_{i_2},$$

which contradicts *ERLB*. So, for any $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

Thus, (A^0) is valid.

The (B^0) ***Part***.

Let $v = (v_I^1, v_O^0) \in V^n$. When $\#I = 0$ or 1, we have already shown as the preliminary results. So, consider the case of $\#I > 1$.

Let $i, j \in I$. From (5), it follows that

$$f_i(v_{\{i,j\}}^1, v_{O\cup I\setminus\{i,j\}}^0) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_i^1).$$

For any $h \neq i, j$, it holds that

$$s_h(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}}) = 0.$$

By *ETE* and feasibility of transfer, for any $h \neq i, j$, it also follows that

$$t_h(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}}) \leq \frac{1}{n} v^1_i.$$

We claim that $t_h(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}}) = \frac{1}{n} v^1_i$. Suppose to the contrary that

$$t_h(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}}) < \frac{1}{n} v^1_i.$$

By *wDE*, for any $\hat{v}_h \in V$ such that $v^0_h < \hat{v}_h < v^1_i$, it follows that

$$s_h(v^1_{\{i,j\}}, \hat{v}_h, v^0_{O\cup I\setminus\{i,j,h\}}) = 0.$$

By *ETE*, we have

$$s_h(v^1_{\{i,j,h\}}, v^0_{O\cup I\setminus\{i,j,h\}}) = \frac{1}{3}.$$

Then, Myerson's Lemma implies that

$$t_h(v^1_{\{i,j,h\}}, v^0_{O\cup I\setminus\{i,j,h\}}) = t_h(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}}) - \frac{1}{3} v^1_i < -(\frac{1}{3} - \frac{1}{n}) v^1_i,$$

which contradicts *ERLB*. Thus, for any $h \neq i, j$, it holds that

$$t_h(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}}) = \frac{1}{n} v^1_i,$$

that is, $f(v^1_{\{i,j\}}, v^0_{O\cup I\setminus\{i,j\}})$ coincides with the allocation determined by the second price trading rule. By the similar way, we can show that for any $k \in I \setminus \{i, j\}$, $f(v^1_{\{i,j,k\}}, v^0_{O\cup I\setminus\{i,j,k\}})$ coincides with the allocation determined by the second price trading rule. Repeating the same argument, we have (B^0). Thus, the first part is valid.

The Second Part.

Given any integer c such that $2 \leq c \leq n-2$. Before proving (A') and (B'), we show preliminary results. Let $(v^1_I, v_{C'}, v^0_O) \in V^n$ be such that $\#C' = c-1$ and $\#I = 1$. For simplicity of notation, we denote $I = \{1\}$, and v_2 as the highest valuation in C'. Pick up any agent $h \in O$.

Note that by *wDE*, *SG*, and *ETE*, we have

$$s_h(v_h, v^1_I, v_{C'}, v^0_{O\setminus\{h\}}) = \begin{cases} 0 & \text{if } v^0_h \leq v_h \leq v_2, \\ 0 \text{ or } \frac{1}{n} & \text{if } v_2 < v_h < v^1_1, \\ \frac{1}{2} & \text{if } v_h = v^1_1. \end{cases}$$

We claim that for any $v_h \in V$ such that $v_2 < v_h < v_1^1$, we have

$$s_h(v_h, v_I^1, v_{C'}, v^0_{O\setminus\{h\}}) = \frac{1}{n}.$$

Suppose to the contrary that for some $\hat{v}_h \in V$ such that $v_2 < \hat{v}_h < v_1^1$, it holds that

$$s_h(\hat{v}_h, v_I^1, v_{C'}, v^0_{O\setminus\{h\}}) = 0.$$

Since, by the induction hypothesis (A), $f_h(v_I^1, v_{C'}, v_O^0) = (0, \frac{1}{n}v_2)$, by *SP*, we have

$$t_h(\hat{v}_h, v_I^1, v_{C'}, v^0_{O\setminus\{h\}}) = \frac{1}{n}v_2,$$

which contradicts *ERLB*. So, for any $v_h \in V$ such that $v_2 < v_h < v_1^1$, we have

$$s_h(v_h, v_I^1, v_{C'}, v^0_{O\setminus\{h\}}) = \frac{1}{n}.$$

Then, by Myerson's Lemma, we have

$$f_h(v_h, v_I^1, v_{C'}, v^0_{O\setminus\{h\}}) = \begin{cases} (0, \frac{1}{n}v_2) & \text{if } v_h^0 \leq v_h \leq v_2 \\ (\frac{1}{n}, 0) & \text{if } v_2 < v_h < v_1^1 \\ (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_1^1) & \text{if } v_h = v_1^1. \end{cases} \tag{6}$$

The (A') ***Part***.
Let $v = (v_I^1, v_C, v_O^0) \in V^n$ be such that $\#C = c$ and $\#I = 1$. We denote $I = \{i_1\}$, and v_{i_2} as the highest valuation in C, that is, $i_2 \in [2_v]$. We divide the argument into two cases.

The case 1: $\#[2_v] = 1$.
From (6), we have $f_{i_2}(v_I^1, v_C, v_O^0) = (\frac{1}{n}, 0)$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_O^0) = \frac{n-1}{n}.$$

Since, from (6), $f_{i_1}(\hat{v}_{i_1}, v_C, v_O^0) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_O^0) = -\frac{n-2}{n}v_{i_2}.$$

Then, for any $i \neq i_1, i_2$, it holds that

$$s_i(v_I^1, v_C, v_O^0) = 0.$$

By the induction hypothesis (A), for any $k \in C \setminus \{i_2\}$, it follows that

$$f_k(v_I^1, v_{C\setminus\{k\}}, v^0_{O\cup\{k\}}) = (0, \frac{1}{n}v_{i_2}).$$

So, by *SP*, for any $k \in C \setminus \{i_2\}$, it holds that

$$t_k(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

We claim that for any $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

Suppose to the contrary that for some $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) \neq \frac{1}{n}v_{i_2}.$$

If $t_h(v_I^1, v_C, v_O^0) > \frac{1}{n}v_{i_2}$, then, by *ETE*, it violates the feasibility of transfer. So, consider the case of $t_h(v_I^1, v_C, v_O^0) < \frac{1}{n}v_{i_2}$. By *wDE*, *SG*, and *ETE*, we have

$$s_h(\hat{v}_h, v_I^1, v_C, v_{O\setminus\{h\}}^0) = 0,$$

where $\hat{v}_h = v_{i_2}$. Then, by *SP*, it holds that

$$t_h(\hat{v}_h, v_I^1, v_C, v_{O\setminus\{h\}}^0) = t_h(v_I^1, v_C, v_O^0) < \frac{1}{n}v_{i_2},$$

which contradicts *ERLB*. So, for any $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n}v_{i_2}.$$

The case 2: $\#[2_v] > 1$.
From (6), for any $i \in [2_v]$, we have $f_i(v_I^1, v_C, v_O^0) = (0, \frac{1}{n}v_{i_2})$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_O^0) = 1.$$

Since, by the induction hypothesis (B), $f_{i_1}(\hat{v}_{i_1}, v_C, v_O^0) = (\frac{1}{\#[2_v]+1}, -(\frac{1}{\#[2_v]+1} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_O^0) = -\frac{n-1}{n}v_{i_2}.$$

Then, for any $i \notin [1_v] \cup [2_v]$, it holds that

$$s_i(v_I^1, v_C, v_O^0) = 0.$$

By the induction hypothesis (A), for any $k \in C \setminus [2_v]$, it follows that

$$f_k(v_I^1, v_{C\setminus\{k\}}, v_{O\cup\{k\}}^0) = (0, \frac{1}{n}v_{i_2}).$$

So, by *SP*, for any $k \in C \setminus [2_v]$, it holds that

$$t_k(v_I^1, v_C, v_O^0) = \frac{1}{n} v_{i_2}.$$

We claim that for any $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n} v_{i_2}.$$

Suppose to the contrary that for some $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) \neq \frac{1}{n} v_{i_2}.$$

If $t_h(v_I^1, v_C, v_O^0) > \frac{1}{n} v_{i_2}$, then, by *ETE*, it violates the feasibility of transfer. So, consider the case of $t_h(v_I^1, v_C, v_O^0) < \frac{1}{n} v_{i_2}$. By *wDE*, *SG*, and *ETE*, we have

$$s_h(\hat{v}_h, v_I^1, v_C, v_{O \setminus \{h\}}^0) = 0,$$

where $\hat{v}_h = v_{i_2}$. Then, by *SP*, it holds that

$$t_h(\hat{v}_h, v_I^1, v_C, v_{O \setminus \{h\}}^0) = t_h(v_I^1, v_C, v_O^0) < \frac{1}{n} v_{i_2},$$

which contradicts *ERLB*. So, for any $h \in O$, it holds that

$$t_h(v_I^1, v_C, v_O^0) = \frac{1}{n} v_{i_2}.$$

Thus, (A') is valid.

The (B') ***Part***.

Let $v = (v_I^1, v_C, v_O^0) \in V^n$ be such that $\#C = c - 1$ and $I \neq \emptyset$. If $\#I = 1$, then the induction hypothesis (A) implies the conclusion. So, consider the case of $\#I > 1$.

Let $i, j \in I$. Let $i_2 \in C$ be such that his valuation v_{i_2} is the highest in C. From (A'), it holds that

$$f_j(\hat{v}_j, v_i^1, v_C, v_{I \setminus \{i,j\}}^0, v_O^0) = (0, \frac{1}{n} v_{i_2})$$

where $\hat{v}_j = v_{i_2}$, and that for any $v'_j \in V$ such that $v_i^1 > v'_j > v_{i_2}$,

$$s_j(v'_j, v_i^1, v_C, v_{I \setminus \{i,j\}}^0, v_O^0) = \frac{1}{n}.$$

Since, by *wDE* and *ETE*, $s_j(v_{\{i,j\}}^1, v_C, v_{I \setminus \{i,j\}}^0, v_O^0) = \frac{1}{2}$, by Myerson's Lemma, it follows that

$$t_j(v_{\{i,j\}}^1, v_C, v_{I \setminus \{i,j\}}^0, v_O^0) = -(\frac{1}{2} - \frac{1}{n}) v_i^1.$$

Then, for any $k \neq i, j$, we have

$$s_k(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) = 0.$$

Note that by the induction hypothesis (B), for any $k \in C$, it follows that

$$f_k(v^1_{\{i,j\}}, v_{C\setminus\{k\}}, v^0_{I\setminus\{i,j\}}, v^0_{O\cup\{k\}}) = (0, \frac{1}{n}v^1_i).$$

So, by *SP*, for any $k \in C$, we have

$$t_k(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) = \frac{1}{n}v^1_i.$$

We claim that for any $h \in O$, it holds that

$$t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) = \frac{1}{n}v^1_i.$$

Suppose to the contrary that for some $h \in O$, it holds that

$$t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) \neq \frac{1}{n}v^1_i.$$

If $t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) > \frac{1}{n}v^1_i$, then, by *ETE*, it violates the feasibility of transfer. So, consider the case of $t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) < \frac{1}{n}v^1_i$. By *wDE* and *ETE*, we have

$$s_h(v^1_{\{i,j,h\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_{O\setminus\{h\}}) = \frac{1}{3}.$$

Since, by *wDE*, for any $v'_h \in V$ such that $v^1_i > v'_h > v^0_h$,

$$s_h(v'_h, v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_{O\setminus\{h\}}) = 0,$$

by Myerson's Lemma, we have

$$t_h(v^1_{\{i,j,h\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_{O\setminus\{h\}}) = t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) - \frac{1}{3}v^1_i < -(\frac{1}{3} - \frac{1}{n})v^1_i,$$

which contradicts *ERLB*. So, by *ETE*, for any $h \in O \cup I \setminus \{i, j\}$, it holds that

$$t_h(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O) = \frac{1}{n}v^1_i,$$

that is, $f(v^1_{\{i,j\}}, v_C, v^0_{I\setminus\{i,j\}}, v^0_O)$ coincides with the allocation determined by the second price trading rule. By the similar way, we can show that for any $k \in I \setminus \{i, j\}$, $f(v^1_{\{i,j,k\}}, v_C, v^0_{I\setminus\{i,j,k\}}, v^0_O)$ coincides with the allocation determined by the second price trading rule. Repeating the same argument, we have (B'). Thus, the second part is valid. Therefore, this theorem is valid.

□

7.5 Proof of Theorem 5

Let f be a mechanism satisfying *SP*, *EF*, *NTNT*, *ETE*, *wDE*, and *SG*. In the following, for any partition (I, C, X) of N where some set may be empty, we use the notation $v = (v_I^1, v_C, v_X^x)$ in which for any $i, j \in I$, any $k \in C$, and any $h, h' \in X$, $v_i^1 = v_j^1 > v_k > v_h^x = v_{h'}^x$, where v_i^1 and v_h^x are any values in V.

1. (A^0) For any $C \subset N$ such that $\#C = 1$, and any $I \subset N$ such that $\#I = 1$, $f(v_I^1, v_C, v_X^x)$ coincides with the allocation determined by the quasi second price mechanism.
   (B^0) For any $I \subset N$, $f(v_I^1, v_X^x)$ also do.

2. Given any integer c such that $2 \leq c \leq n-1$. If
   (A) for any $C \subset N$ such that $\#C = c-1$, and any $I \subset N$ such that $\#I = 1$, $f(v_I^1, v_C, v_X^x)$ coincides with the allocation determined by the quasi second price mechanism, and
   (B) for any $C \subset N$ such that $\#C = c-2$, and any non-empty $I \subset N$, $f(v_I^1, v_C, v_X^x)$ also do, then
   (A') for any $C \subset N$ such that $\#C = c$, and any $I \subset N$ such that $\#I = 1$, $f(v_I^1, v_C, v_X^x)$ also do, and
   (B') for any $C \subset N$ such that $\#C = c-1$, and any non-empty $I \subset N$, $f(v_I^1, v_C, v_X^x)$ also do.

The First Part.

Pick up any two agents, say $1, 2 \in N$, and set $X = N \setminus \{1, 2\}$. Let $v_1^1, v_1^x \in V$ be such that $v_1^1 > v_1^x$. By *ETE* and *NTNT*, we have for any $i \in N$,

$$f_i(v_{X \cup \{1,2\}}^x) = (\frac{1}{n}, 0).$$

Then, by *wDE*, *SG*, and *ETE*, we have

$$s_1(v_1^1, v_{X \cup \{2\}}^x) = 1.$$

By Myerson's Lemma, we also have

$$t_1(v_1^1, v_{X \cup \{2\}}^x) = -\frac{n-1}{n} v_2^x.$$

Then, by *ETE*, *EF* and feasibility of transfer, it follows that for any $i \neq 1$,

$$f_i(v_1^1, v_{X \cup \{2\}}^x) = (0, \frac{1}{n} v_2^x).$$

So, $f(v_1^1, v_{X\cup\{2\}}^x)$ coincides with the allocation determined by the quasi second price mechanism. By *wDE*, *SG*, and *ETE*, we have

$$s_1(v_{\{1,2\}}^1, v_X^x) = s_2(v_{\{1,2\}}^1, v_X^x) = \frac{1}{2}.$$

By *wDE* and *SG*, for any $\hat{v}_2 \in V$ such that $v_1^x < \hat{v}_2 < v_1^1$, it follows that $s_2(v_1^1, \hat{v}_2, v_X^x)$ is either 0 or $\frac{1}{n}$.

We claim that for any such $\hat{v}_2$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_X^x) = \frac{1}{n}.$$

Suppose to the contrary that for some $\hat{v}_2 \in V$ such that $v_1^x < \hat{v}_2 < v_1^1$, it holds that

$$s_2(v_1^1, \hat{v}_2, v_X^x) = 0.$$

Then, by Myerson's Lemma, there exists $a \in V$ such that $\hat{v}_2 \leq a \leq v_1^1$, and the following is satisfied: For any $v_2' \in V$ such that $v_1^x < v_2' < a$, we have

$$s_2(v_1^1, v_2', v_X^x) = 0$$

and for any $v_2' \in V$ such that $a < v_2' < v_1^1$, we have

$$s_2(v_1^1, v_2', v_X^x) = \frac{1}{n}.$$

Furthermore, Myerson's Lemma gives that

$$t_2(v_{\{1,2\}}^1, v_X^x) = -(\frac{1}{2} - \frac{1}{n})v_1^1 - (a - v_1^x)\frac{1}{n}.$$

If $t_3(v_{\{1,2\}}^1, v_X^x) \geq \frac{1}{n}v_1^1$, then agent 1 envies agent 3. So, it must be

$$t_3(v_{\{1,2\}}^1, v_X^x) < \frac{1}{n}v_1^1.$$

By *wDE*, for any $\hat{v}_3 \in V$ such that $v_1^x < \hat{v}_3 < v_1^1$, it follows that

$$s_3(v_{\{1,2\}}^1, \hat{v}_3, v_{X\setminus\{3\}}^x) = 0.$$

By *ETE*, we have

$$s_3(v_{\{1,2,3\}}^1, v_{X\setminus\{3\}}^x) = \frac{1}{3}.$$

Then, Myerson's Lemma implies that

$$t_3(v_{\{1,2,3\}}^1, v_{X\setminus\{3\}}^x) = t_3(v_{\{1,2\}}^1, v_X^x) - \frac{1}{3}v_1^1 < -(\frac{1}{3} - \frac{1}{n})v_1^1.$$

If $t_4(v^1_{\{1,2,3\}}, v^x_{X\setminus\{3\}}) \geq \frac{1}{n}v^1_1$, then agent 3 envies agent 4. So, it must be

$$t_4(v^1_{\{1,2,3\}}, v^x_{X\setminus\{3\}}) > \frac{1}{n}v^1_1.$$

By repeating the same argument, we have

$$t_n(v^1_{\{1,2,\ldots,n\}}) < -(\frac{1}{n} - \frac{1}{n})v^1_1 = 0.$$

Since, by *ETE*, for any $i \in N$, $s_i(v^1_{\{1,2,\ldots,n\}}) = \frac{1}{n}$, this contradicts *NTNT*. Thus, for any $\hat{v}_2 \in V$ such that $v^x_1 < \hat{v}_2 < v^1_1$, it holds that

$$s_2(v^1_1, \hat{v}_2, v^x_X) = \frac{1}{n}.$$

Then, by Myerson's Lemma, we have

$$f_2(v^1_1, v_2, v^x_X) = \begin{cases} (\frac{1}{2}, -(\frac{1}{2} + \frac{1}{n})v^1_1) & \text{if } v_2 = v^1_1, \\ (\frac{1}{n}, 0) & \text{if } v^1_1 > v_2 > v^x_1. \end{cases} \tag{7}$$

The (A^0) ***Part***.
Let $C \subset N$ be such that $\#C = 1$, and $I \subset N$ be such that $\#I = 1$. Let $v = (v^1_I, v_C, v^x_X) \in V^n$. We denote $I = \{i_1\}$ and $C = \{i_2\}$. From (7), we have $f_{i_2}(v^1_I, v_C, v^x_X) = (\frac{1}{n}, 0)$. Then, by *wDF*, we have

$$s_{i_1}(v^1_I, v_C, v^x_X) = \frac{n-1}{n}.$$

Since, from (7), $f_{i_1}(\hat{v}_{i_1}, v_C, v^x_X) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v^1_I, v_C, v^x_X) = -\frac{n-2}{n}v_{i_2}.$$

Then, for any $h \in X$, it holds that

$$s_h(v^1_I, v_C, v^x_X) = 0.$$

By *ETE* and feasibility of transfer, for any $h \in X$, it also holds that

$$t_h(v^1_I, v_C, v^x_X) \leq \frac{1}{n}v_{i_2}.$$

We claim that $t_h(v^1_I, v_C, v^x_X) = \frac{1}{n}v_{i_2}$. Suppose to the contrary that

$$t_h(v^1_I, v_C, v^x_X) < \frac{1}{n}v_{i_2}.$$

By *wDE*, *SG*, and *ETE*, it follows that

$$s_h(v_I^1, v_C, \hat{v}_h, v_{X\setminus\{h\}}^x) = 0,$$

where $\hat{v}_h = v_{i_2}$. Then, *SP* implies that

$$t_h(v_I^1, v_C, \hat{v}_h, v_{X\setminus\{h\}}^x) = t_h(v_I^1, v_C, v_X^x) < \frac{1}{n}v_{i_2}.$$

For $k \neq i_1, i_2, h$, by *wDE* and *EF*, it follows that

$$t_k(v_I^1, v_C, \hat{v}_h, v_{X\setminus\{h\}}^x) = t_h(v_I^1, v_C, \hat{v}_h, v_{X\setminus\{h\}}^x) < \frac{1}{n}v_{i_2}.$$

By repeating the same argument, we have,

$$t_{h'}(v_I^1, v_C, \hat{v}_X) < \frac{1}{n}v_{i_2},$$

where for any $h' \in X$, $\hat{v}_{h'} = v_{i_2}$. Since $f(v_1^1, \hat{v}_{X\cup\{i_2\}})$ coincides with the allocation determined by the quasi second price mechanism, this is a contradiction. Thus, for any $h \in X$, it holds that

$$t_h(v_I^1, v_C, v_X^x) = \frac{1}{n}v_{i_2}.$$

Thus, (A^0) is valid.

The (B^0) ***Part***.

Let $I \subset N$ be non-empty set. Let $v = (v_I^1, v_X^x) \in V^n$. When $\#I = 1$, we have already shown. So, consider the case of $\#I > 1$.

Let $i, j \in I$. From (7), it follows that

$$f_i(v_{\{i,j\}}^1, v_{X\cup I\setminus\{i,j\}}^x) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_j^1).$$

Then, for any $h \neq i, j$, it follows that

$$s_h(v_{\{i,j\}}^1, v_{X\cup I\setminus\{i,j\}}^x) = 0.$$

By *ETE* and feasibility of transfer, for any $h \neq i, j$, it also follows that

$$t_h(v_{\{i,j\}}^1, v_{X\cup I\setminus\{i,j\}}^x) \leq \frac{1}{n}v_i^1.$$

We claim that $t_h(v_{\{i,j\}}^1, v_{X\cup I\setminus\{i,j\}}^x) = \frac{1}{n}v_i^1$. Suppose to the contrary that

$$t_h(v_{\{i,j\}}^1, v_{X\cup I\setminus\{i,j\}}^x) < \frac{1}{n}v_i^1.$$

By *wDE*, for any $\hat{v}_h \in V$ such that $v_1^x < \hat{v}_h < v_1^1$, it follows that

$$s_h(v^1_{\{i,j\}}, \hat{v}_h, v^x_{X\cup I\setminus\{i,j,h\}}) = 0.$$

By *ETE*, we have

$$s_h(v^1_{\{i,j,h\}}, v^x_{X\cup I\setminus\{i,j,h\}}) = \frac{1}{3}.$$

Then, Myerson's Lemma implies that

$$t_h(v^1_{\{i,j,h\}}, v^x_{X\cup I\setminus\{i,j,h\}}) = t_h(v^1_{\{i,j\}}, v^x_{X\cup I\setminus\{i,j\}}) - \frac{1}{3}v_i^1 < -(\frac{1}{3} - \frac{1}{n})v_i^1.$$

For $k \neq i, j, h$, if $t_k(v^1_{\{i,j,h\}}, v^x_{X\cup I\setminus\{i,j,h\}}) \geq \frac{1}{n}v_i^1$, then agent h envies agent k. So, it must be

$$t_k(v^1_{\{i,j,h\}}, v^x_{X\cup I\setminus\{i,j,h\}}) < \frac{1}{n}v_i^1.$$

By repeating the same argument, we have

$$t_n(v^1_{\{1,2,\ldots,n\}}) < -(\frac{1}{n} - \frac{1}{n})v_i^1 = 0.$$

Since, by *ETE*, for any $i \in N$, $s_i(v^1_{\{1,2,\ldots,n\}}) = \frac{1}{n}$, this contradicts *NTNT*. Thus, for any $h \neq i, j$, it holds that

$$t_h(v^1_{\{i,j\}}, v^x_{X\cup I\setminus\{i,j\}}) = \frac{1}{n}v_i^1,$$

that is, $f(v^1_{\{i,j\}}, v^x_{X\cup I\setminus\{i,j\}})$ coincides with the allocation determined by the quasi second price mechanism. By the same way, we can show that for any $k \in I \setminus \{i,j\}$, $f(v^1_{\{i,j,k\}}, v^x_{X\cup I\setminus\{i,j,k\}})$ coincides with the allocation determined by the quasi second price mechanism. By repeating the same argument, we have (B^0). Thus, the first part is valid.

The Second Part.

Given any integer c such that $2 \leq c \leq n-1$. Let $C' \subset N$ be such that $\#C' = c - 1$. Let $I \subset N$ be such that $\#I = 1$. Let $(v_I^1, v_{C'}, v_X^x) \in V^n$. For simplicity of notation, we denote $I = \{1\}$, and v_2 as the highest valuation in C'. Pick up any agent $h \in X$.

Note that by *wDE*, *SG*, and *ETE*, we have

$$s_h(v_h, v_I^1, v_{C'}, v^x_{X\setminus\{h\}}) = \begin{cases} 0 & \text{if } v_h^x \leq v_h \leq v_2, \\ 0 \text{ or } \frac{1}{n} & \text{if } v_2 < v_h < v_1^1, \\ \frac{1}{2} & \text{if } v_h = v_1^1. \end{cases}$$

We claim that for any $v_h \in V$ such that $v_2 < v_h < v_1^1$, we have

$$s_h(v_h, v_I^1, v_{C'}, v_{X\setminus\{h\}}^x) = \frac{1}{n}.$$

Suppose to the contrary that for some $\hat{v}_h \in V$ such that $v_2 < \hat{v}_h < v_1^1$, it holds that

$$s_h(v_h, v_I^1, v_{C'}, v_{X\setminus\{h\}}^x) = 0.$$

Then, by Myerson's Lemma, there exists $a \in V$ such that $\hat{v}_h \leq a \leq v_1$, and the following is satisfied: For any $v'_h \in V$ such that $v_h^x < v'_h < a$, we have

$$s_h(v'_h, v_I^1, v_{C'}, v_{X\setminus\{h\}}^x) = 0$$

and for any $v'_h \in V$ such that $a < v'_h < v_1^1$, we have

$$s_h(v'_h, v_I^1, v_{C'}, v_{X\setminus\{h\}}^x) = \frac{1}{n}.$$

Since, by the induction hypothesis (A), $f_h(v_I^1, v_{C'}, v_X^x) = (0, \frac{1}{n}v_2)$, Myerson's Lemma also gives that

$$t_h(v_{I\cup\{h\}}^1, v_{C'}, v_{X\setminus\{h\}}^x) = -(\frac{1}{2} - \frac{1}{n})v_1^1 - \frac{1}{n}(a - v_2).$$

Note that, by *wDE*, for any $j \in C'$, $s_j(v_{I\cup\{h\}}^1, v_{C'}, v_{X\setminus\{h\}}^x) = 0$. Since, by the induction hypothesis (B), for any $j \in C'$, $f_j(v_{I\cup\{h\}}^1, v_{C'\setminus\{j\}}, v_{X\cup\{j\}\setminus\{h\}}^x) = (0, \frac{1}{n}v_1^1)$, by strategy-proofness, it holds that

$$t_j(v_{I\cup\{h\}}^1, v_{C'}, v_{X\setminus\{h\}}^x) = \frac{1}{n}v_1^1.$$

These imply that agent h envies agent j. Thus, for any $v_h \in V$ such that $v_2 < v_h < v_1^1$, we have

$$s_h(v_h, v_I^1, v_{C'}, v_{X\setminus\{h\}}^x) = \frac{1}{n}.$$

Then, by Myerson's Lemma, we have

$$f_h(v_h, v_I^1, v_{C'}, v_{X\setminus\{h\}}^x) = \begin{cases} (0, \frac{1}{n}v_2) & \text{if } v_h^x \leq v_h \leq v_2 \\ (\frac{1}{n}, 0) & \text{if } v_2 < v_h < v_1^1 \\ (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_1^1) & \text{if } v_h = v_1^1. \end{cases} \quad (8)$$

***The* (A') *Part*.**

Let $C \subset N$ be such that $\#C = c$, and $I \subset N$ be such that $\#I = 1$. Let

$v = (v_I^1, v_C, v_X^x) \in V^n$. We denote $I = \{i_1\}$, and v_{i_2} as the highest valuation in C, that is, $i_2 \in [2]$. We divide the argument into two cases.

The case 1: $\#[2] = 1$.
From (8), we have $f_{i_2}(v_I^1, v_C, v_X^x) = (\frac{1}{n}, 0)$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_X^x) = \frac{n-1}{n}.$$

Since, from (8), $f_{i_1}(\hat{v}_{i_1}, v_C, v_X^x) = (\frac{1}{2}, -(\frac{1}{2} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_X^x) = -\frac{n-2}{n} v_{i_2}.$$

Then, for any $i \neq i_1, i_2$, it holds that

$$s_i(v_I^1, v_C, v_X^x) = 0.$$

By the induction hypothesis (A), for any $k \in C \setminus \{i_2\}$, it follows that

$$f_k(v_I^1, v_{C\setminus\{k\}}, v_{X\cup\{k\}}^x) = (0, \frac{1}{n} v_{i_2}).$$

So, by *SP*, for any $k \in C \setminus \{i_2\}$, it holds that

$$t_k(v_I^1, v_C, v_X^x) = \frac{1}{n} v_{i_2}.$$

Then, by *EF*, for any $h \in X$, it follows that

$$t_h(v_I^1, v_C, v_X^x) = \frac{1}{n} v_{i_2}.$$

The case 2: $\#[2] > 1$.
From (8), for any $i \in [2]$, we have $f_i(v_I^1, v_C, v_X^x) = (0, \frac{1}{n} v_{i_2})$. Then, by *wDF*, we have

$$s_{i_1}(v_I^1, v_C, v_X^x) = 1.$$

Since, by the induction hypothesis (B), $f_{i_1}(\hat{v}_{i_1}, v_C, v_X^x) = (\frac{1}{\#[\hat{1}]}, -(\frac{1}{\#[\hat{1}]} - \frac{1}{n})v_{i_2})$ where $\hat{v}_{i_1} = v_{i_2}$ and $\#[\hat{1}]$ is at $(\hat{v}_{i_1}, v_C, v_X^x)$, by Myerson's Lemma, we have

$$t_{i_1}(v_I^1, v_C, v_X^x) = -\frac{n-1}{n} v_{i_2}.$$

Then, by *wDE* and *EF*, for any $i \notin [1] \cup [2]$, it holds that

$$f_i(v_I^1, v_C, v_X^x) = (0, \frac{1}{n} v_{i_2}).$$

Thus, (A') is valid.

The (B') ***Part***.

Let $C \subset N$ be such that $\#C = c - 1$, and $I \subset N$ be non-empty set. Let $v = (v_I^1, v_C, v_X^x) \in V^n$. If $\#I = 1$, then the induction hypothesis (A) implies the conclusion. So, consider the case of $\#I > 1$.

Let $i, j \in I$. Let $i_2 \in C$ be such that his valuation v_{i_2} is the highest in C. From (A'), it holds that

$$f_j(\hat{v}_j, v_i^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = (0, \frac{1}{n}v_{i_2})$$

where $\hat{v}_j = v_{i_2}$, and that for any $v_j' \in V$ such that $v_i^1 > v_j' > v_{i_2}$,

$$s_j(v_j', v_i^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = \frac{1}{n}.$$

Since, by *wDE* and *ETE*, $s_j(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = \frac{1}{2}$, by Myerson's Lemma, it follows that

$$t_j(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = -(\frac{1}{2} - \frac{1}{n})v_i^1.$$

Then, for any $k \neq i, j$, we have

$$s_k(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = 0.$$

Note that by the induction hypothesis (B), for any $k \in C$,

$$f_k(v_{\{i,j\}}^1, v_{C\setminus\{k\}}, v_{I\setminus\{i,j\}}^x, v_{X\cup\{k\}}^x) = (0, \frac{1}{n}v_i^1).$$

So, by *SP*, for any $k \in C$, we have

$$t_k(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = \frac{1}{n}v_i^1.$$

Then, by *EF*, for any $h \in X \cup I \setminus \{i, j\}$, it also follows that

$$t_h(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x) = \frac{1}{n}v_i^1,$$

that is, $f(v_{\{i,j\}}^1, v_C, v_{I\setminus\{i,j\}}^x, v_X^x)$ coincides with the allocation determined by the quasi second price mechanism. By the same way, we can show that for any $k \in I \setminus \{i, j\}$, $f(v_{\{i,j,k\}}^1, v_C, v_{I\setminus\{i,j,k\}}^x, v_X^x)$ coincides with the allocation determined by the quasi second price mechanism. By repeating the same argument, we have (B'). Thus, the second part is valid. Therefore, this theorem is valid.

□

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