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An Equivalence of Secure Implementability and Full Implementability in Truthful Strategies in Pure Exchange Economies with Leontief Utility Functions

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An Equivalence of Secure Implementability and Full Implementability in Truthful Strategies in Pure Exchange Economies with Leontief Utility Functions *

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Abstract

This paper investigates the relationship between secure implementability (Saijo, T., T. Sjöström, and T. Yamato (2007) "Secure Implementation," *Theoretical Economics* 2, pp.203-229) and full implementability in truthful strategies (Nicolò, A. (2004) "Efficiency and Truthfulness with Leontief Preferences. A Note on Two-Agent, Two-Good Economies," *Review of Economic Design* 8, pp.373-382) in pure exchange economies with Leontief utility functions. In general, secure implementability is stronger than full implementability in truthful strategies. However, we find that the opposite relationship is established if the social choice function satisfies non-wastefulness (Li, T. and J. Xue (2011) "Egalitarian Division under Leontief Preferences," mimeo). Together with the result of Li and Xue (2011), this relationship implies the existence of desirable and securely implementable social choice functions in pure exchange economies with Leontief utility functions contrary to almost all environments.

Key words: Secure implementability, Full implementability in truthful strategies, Strategy-proofness, Pure exchange economy, Leontief utility function.

JEL classification: C72, D51, D52, D71, D78.

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1 Introduction

1.1 Back Background

In this paper, we study strategy-proof social choice functions in pure exchange economies with $n \geq 2$ agents and $m \geq 2$ divisible goods. **Strategy-proofness** requires that truthful revelation is a weakly dominant strategy for each agent in the direct revelation mechanism associated with the social choice function. [1] This property is standard for non-manipulability.

In pure exchange economies, as Hurwicz (1972) notes, Walrasian allocations are manipulable. Moreover, he shows that there exists no social choice function that satisfies strategy-proofness, Pareto-efficiency, and individual rationality in pure exchange economies with $n = 2$ agents and $m = 2$ divisible goods on classical domains. After his seminal work, such negative results have been established on specific domains: classical domains (Dasgupta, Hammond, and Maskin, 1979; Zhou, 1991; Serizawa, 2002; Serizawa and Weymark, 2003), Cobb-Douglas domains (Hashimoto, 2008), CES domains (Ju, 2003), and linear domains (Schummer, 1997). [2] The only exceptions are **Leontief domains**. Nicolò (2004) and Li and Xue (2011) demonstrate that certain social choice functions do satisfy strategy-proofness, Pareto efficiency, and individual rationality on Leontief domains. The above mentioned results illustrate how difficult it is to find strategy-proof social choice functions with desirable properties in pure exchange economies.

Although strategy-proofness is a desirable property, it has one shortcoming; the strategy-proof direct revelation mechanism might have a Nash equilibrium that induces a non-optimal outcome. This problem is solved by **secure implementation** (Saijo, Sjöström, and Yamato, 2007) that is identical with double implementation in dominant strategy equilibria and Nash equilibria. [3] This concept is considered to be a benchmark for constructing mechanisms that work well in laboratory experiments. [4] The possibility of secure implementation has been studied in several environments: voting environments (Saijo, Sjöström, and Yamato, 2007; Berga and Moreno, 2009), public good economies (Saijo, Sjöström, and Yamato, 2007; Nishizaki, 2011), the problems of providing a divisible and private good with monetary transfers (Saijo, Sjöström, and Yamato, 2007; Kumar, 2009), the problems of allocating indivisible and private goods with monetary transfers (Fujinaka and Wakayama, 2008), queueing problems (Nishizaki, forthcoming), Shapley-Scarf housing markets (Fujinaka and Wakayama, 2011), and allotment economies with single-peaked preferences (Bochet and Sakai, 2010). [5] These studies illustrate the difficulty of finding securely implementable social choice functions with desirable properties.

[1] A social choice function is a function that associates an outcome with agents' private information. A direct revelation mechanism associated with a social choice function is a mechanism in which (i) the set of strategy profiles is equivalent to the domain and (ii) the game form is equivalent to the function. For strategy-proofness, see Barberà (2010) in social choice theory and Jackson (2001, 2003) and Maskin and Sjöström (2002) in mechanism design theory.

[2] Barberà and Jackson (1995) abandon Pareto-efficiency and study strategy-proof and individually rational social choice functions.

[3] See Mizukami and Wakayama (2007) and Saijo, Sjöström, and Yamato (2007) for dominant strategy implementation and Maskin (1977), Repullo (1987), and Saijo (1988) for Nash implementation.

[4] See Cason, Saijo, Sjöström, and Yamato (2006) for experimental results on secure implementation.

[5] See also Saijo, Sjöström, and Yamato (2003) for theoretical results on secure implementation.

1.2 Motivation

Almost all previous studies show negative results on secure implementation; there rarely exists a non-trivial and securely implementable social choice function. Such results prompt the investigation into environments in which non-trivial and securely implementable social choice functions exist. In this paper, we investigate pure exchange economies with Leontief utility functions.

Previous studies demonstrate that strategy-proofness is weaker on Leontief domains than other domains. In fact, Nicolò (2004) and Li and Xue (2011) show not only strategy-proof, Pareto efficient, and individually rational social choice functions but also more non-manipulable social choice functions with desirable properties. Note that such non-manipulability is necessary for secure implementation. On the basis of these results, we examine the possibility of secure implementation in pure exchange economies with Leontief utility functions.

In addition, practical interests prompt our investigation. It is natural to assume that each agent has a Leontief utility function in computer science literature. [6] For example, we can consider the problem of allocating some resources (CPU, memory, I/O resources, and so forth) in cloud computing systems. Each user demands such resources to do their jobs. The key point is that each user demands them in a fixed-proportion according to their jobs.

1.3 Related Literature

This paper is most closely related to those of Saijo, Sjöström, and Yamato (2007), Nicolò (2004), and Li and Xue (2011).

Saijo, Sjöström, and Yamato (2007) introduce secure implementation and show a necessary and sufficient condition for secure implementability. [7] They study the possibility of secure implementation in specific environments. In standard quasi-linear environments, they show that Groves-Clarke mechanisms (Clarke, 1971; Groves, 1973) are securely implementable if the private or non-excludable public goods are divisible but not if the goods are not divisible. In single-peaked voting environments, they show that only dictatorial social choice functions are securely implementable. [8]

Nicolò (2004) and Li and Xue (2011) study pure exchange economies with Leontief utility functions. Nicolò (2004) introduces **full implementation in truthful strategies**, that is stronger than dominant strategy implementation but weaker than secure implementation, and characterizes a set of social choice functions that are fully implementable in truthful strategies in addition to certain desirable properties. [9] Note that this characterization is established in environments in which there exist $n = 2$ agents with private endowments and $m = 2$ divisible goods. Li and Xue (2011) characterize such social choice functions in environments with $n \geq 2$ agents and $m \geq 2$ divisible goods, assuming that the resources are socially endowed. To characterize them, they impose **non-wastefulness** (Li and Xue, 2011). If an agent has a Leontief utility function, then a certain amount of specific goods may be redundant to maintain her utility level. In such cases, it is better to transfer the amount of goods to other agents who demand them

[6]See Ghodsi, Zaharia, Hindman, Konwinski, Shenker, and Stoica (2011) for Leontief utility functions and computer science.

[7]See Mizukami and Wakayama (2008) for their alternative characterization of securely implementable social choice functions in terms of a stronger version of Maskin monotonicity (Maskin, 1977).

[8]Note that they require dictatorship on the range of the social choice function.

[9]See Berga and Moreno (2009) for their alternative characterization of minmax rules in terms of full implementability in truthful strategies in single-peaked voting environments.

outside the economy. Non-wastefulness requires such transfers and is incompatible with the balanceness of goods, that Nicolò (2004) imposes on his characterization. Imposing non-wastefulness strongly constrains manipulation and enables more desirable social choice functions that are fully implementable in truthful strategies.

1.4 Our Result

Our main result shows that the **rectangular property** (Saijo, Sjöström, and Yamato, 2007) is weaker than **strong non-bossiness** (Ritz, 1983; 1985) if the social choice function satisfies strategy-proofness and non-wastefulness in pure exchange economies with Leontief utility functions. [10] The rectangular property is a necessary condition for secure implementation. On the other hand, strong non-bossiness is a necessary condition for full implementation in truthful strategies. Because the rectangular property is generally stronger than strong non-bossiness, our main result implies that secure implementability is equivalent to full implementability in truthful strategies if the social choice function satisfies non-wastefulness in pure exchange economies with Leontief utility functions. Together with the result of Li and Xue (2011), this relationship implies the existence of desirable and securely implementable social choice functions contrary to almost all environments.

This paper is organized according to the following sections. Section 2 introduces our model. We define properties of social choice functions related to secure implementability, full implementability in truthful strategies, and non-wastefulness in Section 3. Certain preliminary results on the properties are shown in Section 4. In Section 5, we present our main result. Section 6 concludes this paper.

2 Model

Similar to Li and Xue (2011), we consider pure exchange economies with $n \geq 2$ agents and $m \geq 2$ divisible goods. Let $I \equiv \{1, \ldots, n\}$ be the set of **agents** and $K \equiv \{1, \ldots, m\}$ be the set of **goods**. Let $\mathbb{R}_+ \equiv \{r \in \mathbb{R} \mid r \geq 0\}$. For each $k \in K$, let $e_k \in \mathbb{R}_+$ be the **social endowment of good *k***. For each $i \in I$ and each $k \in K$, let $x_{ik} \in \mathbb{R}_+$ be **consumption of good *k* for agent *i*** and $x_i \equiv (x_{ik})_{k \in K} \in \mathbb{R}^m_+$ be **consumption for agent *i***. For each $i \in I$, $x_i = 0$ means that agent i consumes no goods. Let $x \equiv (x_i)_{i \in I} \in \mathbb{R}^{mn}_+$ be an **allocation** and

$$X \equiv \left\{ x \in \mathbb{R}^{mn}_+ \;\middle|\; \sum_{i \in I} x_{ik} \leq e_k \text{ for each } k \in K \right\}$$

be the set of **feasible allocations**.

A preference of an agent is represented by a Leontief utility function. For each $i \in I$, let $u_i \colon \mathbb{R}^m_+ \to \mathbb{R}$ be a **Leontief utility function for agent *i***; there exists $(\lambda_{ik})_{k \in K} \in \mathbb{R}^m_{++}$ with $\sum_{k \in K} \lambda_{ik} = 1$ such that for each $x_i \in \mathbb{R}^m_+$,

$$u_i(x_i) = \min_{k \in K} \left\{ \frac{x_{ik}}{\lambda_{ik}} \right\},$$

where $\mathbb{R}_{++} \equiv \{r \in \mathbb{R} \mid r > 0\}$. For each $i \in I$, let U_i be the set of utility functions for agent i. Let $u \equiv (u_i)_{i \in I}$ be a profile of utility functions and $U \equiv \prod_{i \in I} U_i$ be the set of profiles of utility functions. For

[10] Strong non-bossiness is called non-corruptibility by Ritz (1983, 1985) and non-bossiness by Saijo, Sjöström, and Yamato (2007).

each $i \in I$, let $u_{-i} \equiv (u_j)_{j \in I \setminus \{i\}}$ be a profile of utility functions other than agent i and $U_{-i} \equiv \prod_{j \in I \setminus \{i\}} U_j$ be the set of profiles of utility functions other than agent i. For each $I_1, I_2 \subseteq I$ with $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$ and each $u, u' \in U$, let (u_{I_1}, u'_{I_2}) be the profile of utility functions in which agent $i \in I_1$ has u_i and agent $i \in I_2$ has u'_i.

A social choice function associates a desirable allocation with a profile of utility functions. Let $f\colon U \to X$ be a **social choice function**. For each $u \in U$ and each $i \in I$, let $f_i(u)$ be the consumption for agent i at the allocation $f(u)$.

Remark 1. Li and Xue (2011) assume that each social choice function inherently satisfies non-wastefulness that is defined in Section 3 (see Definition 4).

Remark 2. Our main result is established even if an agent has a "generalized" Leontief utility function (see Section 5, Remark 6). [11] For each $i \in I$ and each $k \in K$, let $\varphi_{ik}\colon \mathbb{R}_+ \to \mathbb{R}$ be a strictly increasing function. For each $i \in I$ and each $u_i \in U_i$, u_i is a **generalized Leontief utility function for agent *i*** if and only if there exists $(\varphi_{ik})_{k \in K}$ such that for each $x_i \in \mathbb{R}^m_+$,

$$u_i(x_i) = \min_{k \in K} \{\varphi_{ik}(x_{ik})\}.$$

3 Properties of Social Choice Functions

This paper investigates the relationship between **secure implementability** (Saijo, Sjöström, and Yamato, 2007) and **full implementability in truthful strategies** (Nicolò, 2004) in pure exchange economies with Leontief utility functions. In this section, we define the associated properties.

3.1 Secure Implementability and Full Implementability in Truthful Strategies

Saijo, Sjöström, and Yamato (2007) introduce secure implementation that is identical with double implementation in dominant strategy equilibria and Nash equilibria. They show that the social choice function is **securely implementable** if and only if it satisfies **strategy-proofness** and the **rectangular property** (Saijo, Sjöström, and Yamato, 2007).

Strategy-proofness requires that truthful revelation is a weakly dominant strategy for each agent in the direct revelation mechanism associated with the social choice function.

Definition 1. The social choice function f satisfies **strategy-proofness** if and only if for each $u, u' \in U$ and each $i \in I$, $u_i(f_i(u_i, u'_{-i})) \geq u_i(f_i(u'_i, u'_{-i}))$.

The rectangular-property requires that if each agent cannot change her utility by her manipulation, then the outcome cannot change by the all agents' manipulation in the direct revelation mechanism associated with the social choice function.

Definition 2. The social choice function f satisfies the **rectangular property** if and only if for each $u, u' \in U$, if $u_i(f_i(u_i, u'_{-i})) = u_i(f_i(u'_i, u'_{-i}))$ for each $i \in I$, then $f(u) = f(u')$.

[11] See Nicolò (2004) and Li and Xue (2011) for generalized Leontief utility functions in pure exchange economies.

Nicolò (2004) introduces full implementation in truthful strategies, that is stronger than dominant strategy implementation but weaker than secure implementation. By definition, we know that the social choice function is **fully implementable in truthful strategies** if and only if it satisfies **strategy-proofness** and **strong non-bossiness** (Ritz, 1983; 1985). [12]

Strong non-bossiness requires that each agent cannot change the outcome by her manipulation while maintaining her utility level in the direct revelation mechanism associated with the social choice function.

Definition 3. The social choice function f satisfies **strong non-bossiness** if and only if for each $u, u' \in U$ and each $i \in I$, if $u_i(f_i(u_i, u'_{-i})) = u_i(f_i(u'_i, u'_{-i}))$, then $f(u_i, u'_{-i}) = f(u'_i, u'_{-i})$.

By definition, strong non-bossiness is stronger than non-bossiness (Satterthwaite and Sonnenschein, 1981). [13] If $n = 2$, $m = 2$, and each agent has a private endowment, then Nicolò (2004) characterizes a set of social choice functions that are fully implementable in truthful strategies in addition to certain desirable properties.

Saijo, Sjöström, and Yamato (2007) show that the rectangular property is stronger than strong non-bossiness in general environments. [14] This finding implies that secure implementability is generally stronger than full implementability in truthful strategies. However, the opposite relationship is established if the social choice function satisfies **non-wastefulness** (Li and Xue, 2011) in pure exchange economies with Leontief utility functions.

3.2 Non-Wastefulness

If an agent has a Leontief utility function, then a certain amount of specific goods may be redundant to maintain her utility level. In such cases, it is better to transfer the amount of goods to other agents who demand them outside the economy. Li and Xue (2011) define such desirability as non-wastefulness.

For each $i \in I$ and each $u_i \in U_i$, let

$$\gamma(u_i) \equiv \{x_i \in \mathbb{R}^m_+ \mid \text{there exists } t \in \mathbb{R}_+ \text{ such that } x_i = (t\lambda_{ik})_{k \in K}\}$$

be the **critical set for agent i with u_i**, that is, the ray that consists of the entire kink points of u_i beginning at the origin. Note that for each $i \in I$ and each $u_i, u'_i \in U_i$ with $u_i \neq u'_i$, $\gamma(u_i) \neq \gamma(u'_i)$ and $\gamma(u_i) \cap \gamma(u'_i) = \{0\}$.

Remark 3. If an agent has a generalized Leontief utility function, then the critical set may be a curve and intersect with other sets at some points other than the origin.

Non-wastefulness requires that the consumption for each agent associated by the social choice function is in her critical set.

Definition 4. The social choice function f satisfies **non-wastefulness** if and only if for each $u \in U$ and each $i \in I$, $f_i(u) \in \gamma(u_i)$.

[12] See Mizukami and Wakayama (2007) and Saijo, Sjöström, and Yamato (2007) for dominant strategy implementable social choice functions.

[13] The social choice function f satisfies **non-bossiness** if and only if for each $u, u' \in U$ and each $i \in I$, if $f_i(u_i, u'_{-i}) = f_i(u'_i, u'_{-i})$, then $f(u_i, u'_{-i}) = f(u'_i, u'_{-i})$.

[14] See Proposition 2 in Saijo, Sjöström, and Yamato (2007). In addition, see their alternative characterization of securely implementable social choice functions in terms of strong non-bossiness.

Remark 4. Non-wastefulness is incompatible with the balanceness of goods because it requires transferring the redundant amount of specific goods to other agents outside the economy. The balanceness of goods requires that for each $u \in U$ and each $k \in K$, $\sum_{i \in I} f_{ik}(u) = e_k$, where $f_{ik}(u)$ is the consumption of good k for agent $i \in I$ at the allocation $f(u)$.

4 Preliminary Results

In this section, we present certain preliminary results on full implementability in truthful strategies and non-wastefulness in pure exchange economies with Leontief utility functions. Note that full implementability in truthful strategies is equivalent to the combination of strategy-proofness and strong non-bossiness.

4.1 Full Implementability in Truthful Strategies

Lemma 1 shows the uniqueness of each agent's utility maximizer that she can induce if the social choice function satisfies strategy-proofness and strong non-bossiness.

Lemma 1. *Suppose that the social choice function f satisfies **strategy-proofness** and **strong non-bossiness**. For each $u, u' \in U$ and each $i \in I$, if $f_i(u_i, u'_{-i}) \neq f_i(u'_i, u'_{-i})$, then $u_i(f_i(u_i, u'_{-i})) > u_i(f_i(u'_i, u'_{-i}))$.*

Proof. To the contrary, we suppose that there exist $u, u' \in U$ and $i \in I$ such that $f_i(u_i, u'_{-i}) \neq f_i(u'_i, u'_{-i})$ and $u_i(f_i(u_i, u'_{-i})) \leq u_i(f_i(u'_i, u'_{-i}))$. If $u_i(f_i(u_i, u'_{-i})) < u_i(f_i(u'_i, u'_{-i}))$, then we have a contradiction to **strategy-proofness**. If $u_i(f_i(u_i, u'_{-i})) = u_i(f_i(u'_i, u'_{-i}))$, then we find that $f(u_i, u'_{-i}) = f(u'_i, u'_{-i})$ by **strong non-bossiness**. This is a contradiction because $f_i(u_i, u'_{-i}) \neq f_i(u'_i, u'_{-i})$. $\square$

Lemma 1 implies Corollary 1; each agent who consumes no goods cannot consume any goods by her revelation if the social choice function satisfies strategy-proofness and strong non-bossiness.

Corollary 1. *Suppose that the social choice function f satisfies **strategy-proofness** and **strong non-bossiness**. For each $u, u' \in U$ and each $i \in I$, if $f_i(u_i, u'_{-i}) = 0$, then $f_i(u'_i, u'_{-i}) = 0$.*

Proof. To the contrary, we suppose that there exist $u, u' \in U$ and $i \in I$ such that $f_i(u_i, u'_{-i}) = 0$ and $f_i(u'_i, u'_{-i}) \neq 0$. These imply that $f_i(u_i, u'_{-i}) \neq f_i(u'_i, u'_{-i})$. By Lemma 1, we find that $u_i(f_i(u_i, u'_{-i})) > u_i(f_i(u'_i, u'_{-i}))$. This is a contradiction because $f_i(u'_i, u'_{-i}) \neq 0$ by the definition of utility functions. $\square$

4.2 Non-Wastefulness

Lemma 2 shows that each agent's non-zero consumption corresponds one-to-one with her utility function if the social choice function satisfies non-wastefulness.

Lemma 2. *Suppose that the social choice function f satisfies **non-wastefulness**. For each $u, u' \in U$ and each $i \in I$, if $f_i(u_i, u'_{-i}) \neq 0$ and $f_i(u_i, u'_{-i}) = f_i(u'_i, u'_{-i})$, then $u_i = u'_i$.*

Proof. To the contrary, we suppose that there exist $u, u' \in U$ and $i \in I$ such that $f_i(u_i, u'_{-i}) \neq 0$, $f_i(u_i, u'_{-i}) = f_i(u'_i, u'_{-i})$, and $u_i \neq u'_i$. Because $u_i \neq u'_i$, we know that $\gamma(u_i) \cap \gamma(u'_i) = \{0\}$ by the definition of γ. This is a contradiction because $f_i(u_i, u'_{-i}) \neq 0$ and $f_i(u_i, u'_{-i}) = f_i(u'_i, u'_{-i})$. $\square$

Remark 5. Lemma 2 is not established if the agent has a generalized Leontief utility function because the critical set may be a curve and intersect with other sets at some points other than the origin, that is, $\gamma(u_i) \cap \gamma(u'_i) \supset \{0\}$ for some $i \in I$ and some $u_i, u'_i \in U_i$ with $u_i \neq u'_i$.

5 Main Result

As previously stated, the rectangular property is generally stronger than strong non-bossiness. The following theorem shows that the opposite relationship is established if the social choice function satisfies strategy-proofness and non-wastefulness in pure exchange economies with Leontief utility functions.

Theorem. *Suppose that the social choice function satisfies* ***strategy-proofness*** *and* ***non-wastefulness****. If the social choice function satisfies* ***strong non-bossiness****, then it satisfies the* ***rectangular property****.*

Proof. Let f be a social choice function that satisfies strategy-proofness, non-wastefulness, and strong non-bossiness. Let $u, u' \in U$ be such that $u_i(f_i(u_i, u'_{-i})) = u_i(f_i(u'_i, u'_{-i}))$ for each $i \in I$. By **strong non-bossiness**, this implies that

$$f(u_i, u'_{-i}) = f(u'_i, u'_{-i}) \text{ for each } i \in I. \tag{1}$$

Let $I_1 \equiv \{i \in I \mid f_i(u_i, u'_{-i}) = 0\}$ and $I_2 \equiv \{i \in I \mid f_i(u_i, u'_{-i}) \neq 0\}$. Without loss of generality, let $1, 2 \in I_1$. By (1), we know that $f(u_1, u'_{I_1 \setminus \{1\}}, u'_{I_2}) = f(u'_{I_1}, u'_{I_2})$ and $f_2(u'_{I_1}, u'_{I_2}) = 0$. These imply that

$$f_2(u_1, u'_2, u'_{I_1 \setminus \{1,2\}}, u'_{I_2}) = 0. \tag{2}$$

By (2) and Corollary 1, we find that

$$f_2(u_1, u_2, u'_{I_1 \setminus \{1,2\}}, u'_{I_2}) = 0. \tag{3}$$

By (2), (3), and **non-bossiness**, we find that $f(u_1, u_2, u'_{I_1 \setminus \{1,2\}}, u'_{I_2}) = f(u_1, u'_2, u'_{I_1 \setminus \{1,2\}}, u'_{I_2})$. [15] By sequentially replacing u'_i by u_i for each $i \in I_1$ in this manner, we find that

$$f(u_{I_1}, u'_{I_2}) = f(u'). \tag{4}$$

By (1) and Lemma 2, we find that $u_i = u'_i$ for each $i \in I_2$. This implies that

$$f(u) = f(u_{I_1}, u'_{I_2}). \tag{5}$$

By (4) and (5), we find that $f(u) = f(u')$. □

Our theorem is tight. Nicolò (2004) characterizes social choice functions that satisfy strategy-proofness, strong non-bossiness, and the balanceness of goods. Because the balanceness of goods is incompatible with non-wastefulness, they do not satisfy non-wastefulness. By applying the result of Saijo, Sjöström, and Yamato (2007) in single-peaked voting environments, we find that they do not satisfy the rectangular property. [16] This finding implies that non-wastefulness is necessary for our theorem. The following example shows that strategy-proofness is necessary for our theorem.

[15] Note that strong non-bossiness implies non-bossiness.

[16] Note that Nicolò (2004) uses the techniques in single-peaked voting environments to characterize them.

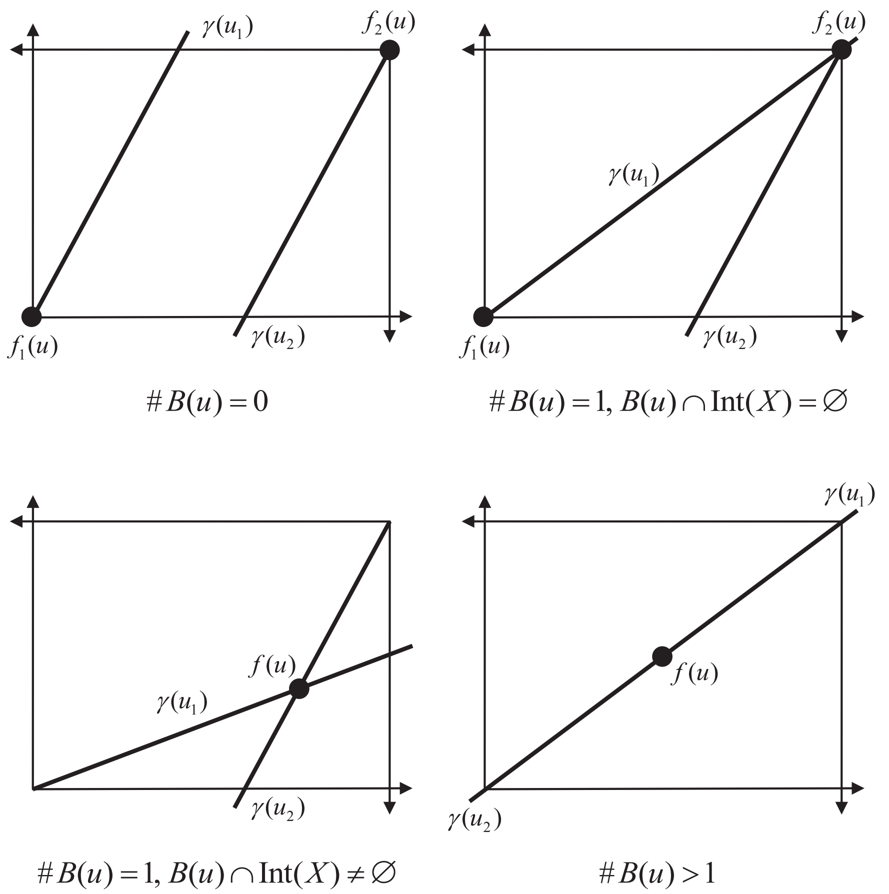

Figure 1: Allocation of f in Example

Example. Suppose that $I = \{1,2\}$ and $K = \{1,2\}$. For each $u \in U$, let $B(u) \equiv \{x \in \gamma(u_1) \times \gamma(u_2) \mid x_{1k} + x_{2k} = e_k$ for each $k \in K\}$. Let f be the social choice function such that for each $u \in U$,

$$f(u) = \begin{cases} (0,0) & \text{if } \#B(u) = 0, \\ (0,0) & \text{if } \#B(u) = 1 \text{ and } B(u) \cap \mathrm{Int}(X) = \emptyset, \\ x & \text{if } \#B(u) = 1 \text{ and } B(u) \cap \mathrm{Int}(X) \neq \emptyset, \\ ((e_1/2, e_2/2), (e_1/2, e_2/2)) & \text{if } \#B(u) > 1, \end{cases}$$

where $x \in B(u) \cap \mathrm{Int}(X)$, $\#B(u)$ means the cardinality of $B(u)$, and $\mathrm{Int}(X)$ means the interior of X (see Figure 1). Obviously, f satisfies non-wastefulness but not strategy-proofness.

We show that f satisfies strong non-bossiness. Let $u, u' \in U$ and $i \in I$ be such that $u_i(f_i(u_i, u'_{-i})) = u_i(f_i(u'_i, u'_{-i}))$. By the definition of f, this implies that $f_i(u_i, u'_{-i}) = f_i(u'_i, u'_{-i})$ and $f_j(u_i, u'_{-i}) = f_j(u'_i, u'_{-i})$ for $j \neq i$. Thus, we find that $f(u_i, u'_{-i}) = f(u'_i, u'_{-i})$. This finding implies that f satisfies strong non-bossiness.

We show that f does not satisfy the rectangular property. Let $u, u' \in U$ be such that $\#B(u) > 1$ and $\#B(u') = 0$. This implies that $f_i(u_i, u'_{-i}) = f_i(u'_i, u'_{-i}) = 0$ for each $i \in I$. Thus, we find that $u_i(f_i(u_i, u'_{-i})) = u_i(f_i(u'_i, u'_{-i}))$ for each $i \in I$ and $f(u) \neq f(u')$. This finding implies that f does not

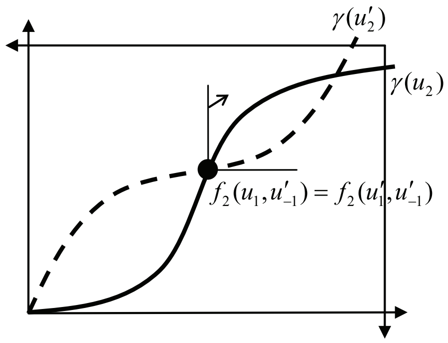

Figure 2: Generalization of Theorem

satisfy the rectangular property.

Remark 6. Our theorem is established even if an agent has a generalized Leontief utility function. The key point is the existence of agents in I_2. Because Corollary 1 is independent of the property of the critical sets, there is no problem in replacing utility functions for agents in I_1 even if they have generalized Leontief utility functions. However, there may be a problem in replacing utility functions for agents in I_2 because Lemma 2 is dependent on the property of the critical sets as we stated in Section 4, Remark 5. Strategy-proofness solves this problem.

Per (1) in our theorem, let $u, u' \in U$ be such that $f(u_i, u'_{-i}) = f(u'_i, u'_{-i})$ for each $i \in I$. Without loss of generality, let $1, 2 \in I_2$ and $f(u_1, u'_{-1}) = f(u'_1, u'_{-1})$. Note that $f_2(u'_2, u'_{-2}) \in \gamma(u_2)$, that is, $\gamma(u_2)$ intersects with $\gamma(u'_2)$ at $f_2(u'_2, u'_{-2})$ because $f(u_i, u'_{-i}) = f(u'_i, u'_{-i})$ for each $i \in I$ (see Figure 2). By non-wastefulness, we know that agent 2's consumption assigned by f is in her critical set. This implies that $f_2(u_1, u_2, u'_{I_1}, u'_{I_2\setminus\{1,2\}})$ is in $\gamma(u_2)$. By strategy-proofness, this implies that $f_2(u_1, u_2, u'_{I_1}, u'_{I_2\setminus\{1,2\}}) = f_2(u_1, u'_2, u'_{I_1}, u'_{I_2\setminus\{1,2\}})$. [17] By non-bossiness, this implies that $f(u_1, u_2, u'_{I_1}, u'_{I_2\setminus\{1,2\}}) = f(u_1, u'_2, u'_{I_1}, u'_{I_2\setminus\{1,2\}})$. By sequentially replacing u'_i by u_i for each $i \in I_2$ in this manner, we find that $f(u'_{I_1}, u_{I_2}) = f(u')$. Together with the argument for agents in I_1, this finding suggests that our theorem is established even if an agent has a generalized Leontief utility function. Note that the tightness is confirmed by similar arguments in the case of Leontief utility functions.

As previously stated, secure implementability is equivalent to the combination of strategy-proofness and the rectangular property. On the other hand, full implementability in truthful strategies is equivalent to the combination of strategy-proofness and strong non-bossiness. Because our theorem suggests that the rectangular property and strong non-bossiness are equivalent for strategy-proof and non-wasteful social choice function, the following corollary is established in pure exchange economies with Leontief utility functions.

Corollary 2. *Suppose that the social choice function satisfies* ***non-wastefulness****. The social choice function is* ***securely implementable*** *if and only if it is* ***fully implementable in truthful strategies****.*

In our economies, Li and Xue (2011) demonstrate social choice functions that satisfy full implementability in truthful strategies and non-wastefulness in addition to certain desirable properties. [18]

[17] This is an advantage of imposing non-wastefulness that makes strategy-proofness weaker.

[18] See Theorem 1 and Remark 3 in Li and Xue (2011).

Together with their demonstration, Corollary 2 implies the existence of desirable and securely implementable social choice functions in our economy.

6 Conclusion

This paper investigates the relationship between secure implementability and full implementability in truthful strategies in pure exchange economies with Leontief utility functions. In general, secure implementability is stronger than full implementability in truthful strategies. However, we find that the opposite relationship is established if the social choice function satisfies non-wastefulness. Together with the result of Li and Xue (2011), this relationship implies the existence of desirable and securely implementable social choice functions contrary to almost all environments. Because of the observations of Cason, Saijo, Sjöström, and Yamato (2006), this relationship suggests a possibility of constructing desirable mechanisms that work well in real-life pure exchange economies with Leontief utility functions. This possibility promotes the study of secure implementation from the perspectives of both theory and experiment.

The existence of desirable and securely implementable social choice functions is dependent on the result of Li and Xue (2011). Because secure implementability is very weak in pure exchange economies with Leontief utility functions, there exist many securely implementable social choice functions. Further research is needed to characterize such social choice functions in different directions from that of Li and Xue (2011) and study the domain-richness in pure exchange economies, as do Fujinaka and Wakayama (2008) in the problems of allocating indivisible and private goods with monetary transfers and Nishizaki (forthcoming) in queueing problems. These interesting topics remain for our future research.

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