GCOE Discussion Paper Series

Global COE Program

Human Behavior and Socioeconomic Dynamics

Discussion Paper No. 270

EXPOSURE PROBLEM IN MULTI-UNIT AUCTIONS

Hikmet Gunay and Xin Meng

June 2012

GCOE Secretariat
Graduate School of Economics
OSAKA UNIVERSITY
1-7 Machikaneyama, Toyonaka, Osaka, 560-0043, Japan

EXPOSURE PROBLEM IN MULTI-UNIT AUCTIONS

Hikmet Gunay and Xin Meng*
University of Manitoba and SWUFE-RIEM

June 24, 2012

Abstract

We characterize the optimal bidding strategies of local and global bidders for two heterogenous licenses in a multi-unit simultaneous ascending auction. The global bidder wants to win both licenses to enjoy synergies; therefore, she bids more than her stand-alone valuation of a license. This exposes her to the risk of losing money even when she wins all licenses. We determine the optimal bidding strategies in the presence of an exposure problem. By using simulation methods, first, we show the frequency of inefficient allocation in the simultaneous ascending auction. Then, we show that the Vickrey-Clarke-Groves (VCG) mechanism may generate more revenue than the simultaneous ascending auction.

JEL Codes:D44, D82

Keywords: Multi-Unit Auctions, Vickrey Clarke Groves (VCG) mechanism, Exposure Problem, Synergies, Complementarity

*Acknowledgements will be added later.

1 Introduction

In a typical American or Canadian spectrum license auction, hundreds of (heterogenous) licenses are sold simultaneously. Each of these licences gives the spectrum usage right of a geographical area to the winning firm. Some *'local'* firms are interested in winning only specific licenses in order to serve local markets while other *'global'* firms are interested in winning all the licenses in order to serve nationwide.[1] The global firms enjoy synergies if they win all the licenses which gives them an incentive to bid over their stand-alone valuations for some licenses. As a result, there is a risk of incurring losses. Therefore, global bidders lower their bids. This is known as the exposure problem.[2]

In a model simplifying the American and the recent Canadian spectrum license auctions, we derive the optimal bidding strategies of local and global firms in a simultaneous ascending auction (SAA) of two licenses. We mainly focus on the optimal bidding strategies when there is the possibility of an exposure problem (i.e., ex-post loss) since managers of cell-phone companies and policy makers would be interested in such numbers. Through simulations, we determine how frequently the exposure problem occurs. In addition, we decompose the frequency into two cases; the case in which the exposure problem occurs when the global bidder wins only one license, and the case in which the exposure problem occurs when the global bidder wins all licenses.

Exposure problem indicates that the allocation may not be efficient. In fact, we show that allocation may be inefficient with 7.84 per cent of the time for some parameter space. We compare the efficiency and revenue properties of the simultaneous ascending auction with those of the Vickrey-Clarke-Groves (VCG) mechanism when bidders are allowed to bid on packages. VCG is an efficient auction that gives the highest revenue among all incentive compatible, individually rational, efficient auctions. In the literature, there are examples

[1] In the recent Canadian Advanced Wireless Spectrum auction, firms such as Globalive and Rogers were interested in all licenses whereas firms such as Bragg Communication and Manitoba Telecom Services (MTS) were interested in East Coast and Manitoba licenses, respectively.

[2] We will interchangeably use exposure problem as follows. We say that an exposure problem occurred whenever the global bidder incurs a loss ex-post.

which show that VCG mechanism may give extreme low revenue in complete information settings, and low revenue is cited as one of the main reasons why VCG mechanism is not used prevalently (e.g., Ausubel and Milgrom (2006)). We show that VCG mechanism may give higher revenue to the seller for many parameter spaces and various distributions in incomplete information setting. For example, when local bidders win the licenses in the SAA auction and the allocation is inefficient, then VCG auction would give higher revenue for the same private valuations. However, when the global bidder wins the license with an ex-post loss (that is, when exposure problem occurs), then SAA auction's revenue would be higher than the VCG mechanism. In this paper, we also show the frequency of inefficient allocation when simultaneous ascending auction is used.

The multi-unit auction literature generally assumes that global bidders have either equal valuations (Englmaier et. al (2009), Kagel and Levin (2005), Katok and Roth (2004), Albano et. al. (2001), Rosenthal and Wang (1996), and Krishna and Rosenthal (1996)) or very large synergies (Albano et al. (2006)). The spectrum licenses for different geographic areas are not homogenous objects; hence, the equal valuation assumption does not fit the Canadian or the American spectrum license auction. Moreover, in a heterogeneous license environment, bidders may not drop out of both auctions simultaneously. This enables us to analyze bidding behavior in the remaining auction, and hence, the exposure problem/ex-post loss. In addition, unlike the aforementioned multi-unit auction papers above, we allow for moderate synergies, and our focus is on the exposure problem and the comparison of revenue and efficiency properties of the simultaneous ascending auction with those of the VCG auction. In our paper, the global bidder will lower his bid because of the exposure problem; however, their optimal strategy still requires him to bid over his stand alone valuation for at least one license. If he wins this license by receiving a potential loss, then he may need to stay in the other license auction to minimize his loss. Therefore, there are cases in which the ex-post loss may occur even when the bidder wins all the licenses.

Goerre and Lien (2010) assume that the marginal valuation of winning a given number of licenses is the same regardless of the composition of the particular licenses. Hence, they find that the optimal drop out price is the same for both licenses. In our paper, marginal valuations are different; hence, the global bidder has a preference over license A and license B. Specifically, we model situations in which winning the spectrum license for, say, Iowa City is different than winning the license for New York city. Hence, our paper shows that the optimal drop out price is not the same. In addition, through simulations, we show the frequency of inefficient allocation and ex-post loss for the simultaneous ascending auction. Like Goerre and Lien (2010), we find VCG mechanism may give higher revenue than the simultaneous ascending auction. However, we show that VCG mechanism gives higher revenue for cases in which local bidders win licenses inefficiently (in the SAA auction), and SAA auction gives higher revenue for cases in which global bidder wins licenses inefficiently.[3] In other words, because of the ex-post loss possibility, global bidder does not bid too much over the stand alone valuation; hence, local bidders win inefficiently. This lowers the revenue. However, global bidder still over-bids and wins licenses with ex-post loss when local bidders are supposed to win in an efficient outcome. This increases the revenue of SAA auction compared to VCG auction at the expense of global bidder.

2 The Model

There are 2 licenses, license A and B for sale.[4] There are one global bidder who demands both licenses and $m_j = m - 1$ local bidders who demand only license $j = A, B$. We explain in detail the cases with more than one global bidder later in the paepr. Specifically, m_j will denote the number of active local bidders on the auction.[5] Both local bidders and the global bidder have a private stand alone valuation for a single license, v_{ij}, where i and j represent the bidder and the license, respectively. The valuations v_{ij} are drawn from the continuous distribution function $F(v_{ij})$ with support on $[0, 1]$ and probability density function $f(v_{ij})$ which is positive everywhere with the only exception that $f(0) \geq 0$ is allowed. The bidders'

[3]Zheng (2008) is mainly interested in showing that jump-bidding will alleviate the exposure problem. We do not allow jump-bidding as the Canadian and the American spectrum auction has not allowed this.

[4]We use two licenses like Albano et. al. (2001 and 2006), Brusco and Lopomo (2002), Chow and Yavas (2009), and Menucicci (2003).

[5]Allowing different number of local bidders per license will not change our qualitative results.

type, global or local, is publicly known.

We consider a setting where the licenses are auctioned off simultaneously through an ascending multi-unit auction. Each license is auctioned off at a different auction (like Krishna and Rosenthal (1996) but unlike Kagel and Levin (2005)) but at the same time. Prices start from zero for both licenses and increase simultaneously and continuously at the same rate. Bidders choose when to drop out. When only one bidder is left on a given license, the clock stops for that license, and the sole remaining bidder wins the license at the price at which the last bidder dropped out. If there are more than one bidder remaining on the other license, its price will continue to increase. If n bidders drop out at the same price and nobody is left in the auction, then each one of them will win the license with probability $\frac{1}{n}$. This is a zero measure event given the valuations are drawn from a continuous distribution function.

The drop-out decision is irreversible. Once a bidder drops out of bidding for a given license, he cannot bid for this license later.[6] The number of active bidders and the drop-out prices are publicly known. We also assume that there is no budget constraints for the bidders.

We assume that there is a homogeneous positive synergy for the global bidder. Specifically, letting bidder 1 be the global bidder, the global bidder's total valuation, given that it wins two licenses is, $V_1 = v_{1A} + v_{1B} + \alpha$, where the synergy term α is assumed to be strictly positive and public knowledge.[7] His stand-alone valuation of license A or B is given by v_{1A} or v_{1B}. Bidder iA, $i = 2, 3, ...m$ is only interested in license A, and her private valuation is v_{iA}. Bidder iB is only interested in license B, and her private valuation is v_{iB}.[8] A local bidder who is interested in license j participates only in license j auction.

We derive a symmetric perfect Bayesian equilibrium through a series of lemmas that follow. First, we describe the equilibrium strategy of the local bidder.

[6] In the real-world auctions, there is activity rule. If the bidders do not have enough highest standing bids, then the number of licenses they may bid on is decreased (in the next rounds). Hence, when there are two licenses, this translates into an irreversible drop-out.

[7] Public knowledge assumption can be removed, and all results are still valid. We assume public knowledge not to complicate the notation.

[8] We do not assume that $v_{iA} > v_{iB}$ since local firms are different; hence, their efficiency may differ.

Lemma 1 *Each local bidder has a weakly dominant strategy to stay in the auction until the price reaches his stand alone valuation.*

This is a well-known result so we skip the proof.

Now, consider a subgame in which all the local bidder drops out of license B auction, and hence, the global bidder wins license B at the price p_B, which in equilibrium is equal to $p_B = max\{v_{2B}, ..., v_{(m)B}\}$ by lemma 1. Then, as the price for license A increases, the global bidder will compare the payoff from dropping out from license A auction at the clock price p (which is $v_{1B} - p_B$) and the payoff from winning license A at price p (which is $v_{1A} + v_{1B} + \alpha - p_B - p$). The updated optimal drop out price, p_A, is found by equating these two equations: $v_{1A} + v_{1B} + \alpha - p_B - p_A = v_{1B} - p_B \Rightarrow p_A = v_{1A} + \alpha$. This is intuitive since by winning license A, the global bidder will also earn the synergy value. If global bidder wins license A first, the updated optimal drop out price can be found symmetrically, and it is $p_B = v_{1B} + \alpha$. We state this as lemma 2.

Lemma 2 *If the global bidder wins license B (or A) first, then it will stay in license A (or B) auction until the price reaches $v_{1A} + \alpha$ (or $v_{1B} + \alpha$)*

The global bidder will not drop out before the price reaches his minimum of stand-alone valuations. Otherwise, they will lose the chance of winning both licenses and enjoying the synergy. In addition, if the global bidder's average valuation, $\frac{V_1}{2} = \frac{v_{1A}+v_{1B}+\alpha}{2}$, exceeds 1, bidding up to his average valuation will shut out the local bidders since local bidders' stand alone valuation can be at most 1. If α is large enough, this condition will always be satisfied. In such a case, the global bidder always wins both licenses in equilibrium. We summarize these results as lemma 3.

Lemma 3 *a) The global bidder stays in both license auctions at least until the price reaches the minimum of his stand-alone valuations.*

b) If his average valuation is greater than 1, the global bidder's equilibrium strategy is to stay in until the price reaches his average valuation.

To calculate the optimal drop out price for the global bidder, consider first the case in which $v_{1A} > v_{1B}$.[9] The global bidder must compare the payoffs for two cases at each price p as the clock is running: **Case 1** is the payoff from dropping out from license B auction at price p and optimally continuing on license A auction. **Case 2** is the payoff from winning license B at price p and optimally continuing on license A auction.[10] At the beginning of the auction, that is $p = 0$, the second case payoff is higher so the global bidder will start by staying in the auction. We show that the difference between these two cases are monotonic in p ; therefore, there is a unique price that makes the global bidder indifferent between these two cases (assuming that the local bidders are still active).[11] This is the optimal drop out price, p_1^*. We show that this price can be calculated at the beginning of the auction. Note that according to Lemma 3, $p_1^* \geq v_{1B}$, and the optimal updated drop out price for license A, after winning license B at price p, is $v_{1A} + \alpha$.

We denote the expected profit of the global bidder for Case 1 by $E\Pi_1^1$ and his expected profit for Case 2 by $E\Pi_1^2$, respectively.

Let $p_A = \max\{v_{2A}, ..., v_{mA}\}$ be the price the global bidder will pay for the license A, if he wins license A. Payoffs are as follows:

$$E\Pi_1^1 = Max\{0, \int_p^{v_{1A}} (v_{1A} - p_A) g(p_A|p) dp_A\} \tag{1}$$

$$E\Pi_1^2 = \int_p^{Min\{v_{1A}+\alpha,1\}} (V_1 - p - p_A) g(p_A|p) dp_A + \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p) g(p_A|p) dp_A \tag{2}$$

The explanation of equation 1 is as follows. After the global bidder drops out of the auction for license B at p, it becomes just like a local bidder, and hence, will continue to

[9] The other case can be calculated symmetrically.

[10] Assuming that it has not won license A yet, the global bidder will drop out of license B first -or at the same time as A- since $v_{1A} > v_{1B}$. It would not make sense to drop out from license A first -since the prices of both auctions increase at the same rate-, and then continuing on the less valued license B.

[11] Another way of calculating the optimal drop out price would be to maximize the expected payoff. However, this way would be much more complex; especially proving the second order conditions. Hence, we exploit the fact that payoff difference above is monotonic in p. Our results coincide with the literature when the licenses are identical.

stay in the auction for license A until v_{1A}. If he wins, he will pay p_A since the local bidder with highest valuation of license A will drop out last (by Lemma 1). In order to calculate his expected profit, global bidder will be using $G(p_A|p)$ (highest order statistic) which is the distribution function of the local bidders' highest valuation p_A for license A given p. When there are $m-1$ local bidders in license A, the distribution function $G(p_A|p)$ and its density function $g(p_A|p)$ are:

$$G(p_A|p) = (F(p_A|p))^{m-1} = (\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-1} \tag{3}$$

$$g(p_A|p) = (m-1)(\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-2}(\frac{f(p_A)}{\int_p^1 f(v)dv}). \tag{4}$$

The first term of $E\Pi_1^2$ is Firm 1's expected profit of winning both licenses; assuming that he wins license B at the price p. If the highest local bidder's valuation p_A is less than the global bidder's (updated) willingness to pay, $v_{1A}+\alpha$, then the global bidder wins license A and pays p_A. Since $p_A < 1$, we use the minimum function in the upper limit of the first integral. The second term of $E\Pi_1^2$ is Firm 1's expected profit of winning only license B which can happen only if $p_A > v_{1A}+\alpha$. Note that the second term is non-positive by Lemma 3 (which is the exposure problem arising from winning only one license).

In Lemma 4 below, we characterize the global bidder's equilibrium bids. It can be found from $E\Pi_1^1 = E\Pi_1^2$. Note that these payoffs are changing as local bidders bidding for A are dropping out; that is, $m-1$ is changing. Therefore, the lemma below gives the global bidder's (updated) equilibrium drop out price as the local bidders drop out. We show, in the proof of lemma 4, that this updated price increases as local bidders of license A drop out.[12]

[12] Local bidders B dropping out of auction does not affect this optimal drop-out price. Seemingly surprising result arises because of two facts. First, here the assumption is $v_{1A} > v_{1B}$ so the global bidder will drop out from license B first,- as explained in footnote 10. Second, when the global bidder makes the calculation, it compares his payoff of dropping out from license B and continuing on A at the given decision price p, and winning license B and continuing on license A at the given decision price p. In both cases, the number of local bidders B have no effect on the decision. However, number of local bidders A will affect its payoff. If $v_{1B} > v_{1A}$, then ONLY the number of local bidders B will affect the global bidder's decision. We emphasize that local bidders B dropping out from auction affect the game though. If there are fewer local bidders B, it is more likely for the global bidder to win license B before the price reaches its optimal drop out price, and then it will update the new optimal drop out price as $v_{1A}+\alpha$.

Lemma 4 *Suppose that the average valuation of the global bidder is less than 1 and there are $m-1$ local bidders bidding on license A where $m-1 \geq 1$ and there is at least one active local bidder B.*

If $v_{1A} > v_{1B}$, the global bidder[13] *will drop out of license B auction at the unique optimal drop-out price $p_1^* \in [0,1]$ that satisfies $E\Pi_1^1 = E\Pi_1^2$. Moreover,*

a) If $v_{1A}+\alpha < 1$, and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B}-v_{1A}) < 0$, then $p_1^* < v_{1A}$ and the global bidder will stay in license A auction until v_{1A} (after dropping out from license B auction).

b) If $v_{1A}+\alpha < 1$, and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B}-v_{1A}) > 0$, then $p_1^* > v_{1A}$ and the global bidder will also drop out of license A auction at p_1^*.

c) If $v_{1A}+\alpha > 1$, and $\int_{v_{1A}}^{1} G(p_A|p)dp_A + (v_{1B}+\alpha-1) < 0$, then $p_1^* < v_{1A}$ and the global bidder will stay in license A auction until v_{1A} (after dropping out from license B auction).

d) If $v_{1A}+\alpha > 1$, and $\int_{v_{1A}}^{1} G(p_A|p)dp_A + (v_{1B}+\alpha-1) > 0$, then $p_1^* > v_{1A}$ and the global bidder will also drop out of license A auction at p_1^*.

Proof. See the Appendix.

We are ready to summarize our Perfect Bayesian equilibrium.

Proposition 5 *(Perfect Bayesian Equilibrium)*

a) Local bidder of each license will stay in the auction j until price reaches their valuation v_{ij} where $j = \{A, B\}$, $i = \{2, 3, ..m-1\}$.

b) A global bidder active only on license j will bid $v_{1j}+\alpha$, if he won license $-j$ other than license j. He will bid v_{1j} when he did not win license $-j$.

c) When $v_{1A} > v_{1B}$ and the average valuation is less than one, the global bidder who is active on both licenses and facing $m-1$ active local bidders on license A will drop out from license B at the price that equates equations 1 and 2.

d) When $v_{1A} < v_{1B}$ and the average valuation is less than one, the global bidder who is active on both licenses and facing $m-1$ active local bidders on license B will drop out from license A at the price that equates equations 1 and 2 (symmetrically replaced v_{1A} with v_{1B}).

[13] If $v_{1A} < v_{1B}$, then the proposition has to be written symmetrically

e) If the average valuation is greater than one, the global bidder will stay in both auctions until price reaches his average valuation.[14]

f) Out-of-equilibrium-path beliefs: When a bidder drops out, the other bidders will see this as an equilibrium behavior. Hence, any out of equilibrium path beliefs can be used.[15]

At the beginning of the game, each bidder calculates its optimal drop-out price. For local bidders, the optimal drop out prices are their valuations. In equilibrium, it is optimal for the global bidder to stay in the auctions for both licenses up to his optimal drop-out price calculated in Lemma 4. When his average valuation exceeds 1, he will stay until this average valuation and win both licenses. When the price reaches the minimum of these optimal drop-out prices, that bidder drops out of license auction. If, for example, the global bidder wins license A, the global bidder would continue to stay in the auction for license A until the price reaches $v_{1A} + \alpha$. At this price, he finds that the payoff from winning only license B is more than the payoff from winning both licenses even though it will enjoy synergy; hence, it drops out.

The following is a corollary of Lemma 4, and is an example for the optimal drop out price when $F(.)$ is a uniform distribution.

Corollary 6 *Assume that valuations are drawn from a uniform distribution with a support* $[0, 1]$*. In addition, assume that* $v_{1A} > v_{1B}$ *(other case is symmetrically found by exchanging*

[14]Price does not have to stop increasing at 1.

[15]If local bidders drop out, global bidder will think that local bidder is using his/her equilibrium strategy of dropping out at his/her valuation. If the global bidder drops out from license B first (or license A first), the local bidders will think that global bidder is using his/her equilibrium strategy of dropping out from license B since he has a valuation of $v_{1B} > v_{1A}$ (or $v_{1A} > v_{1B}$) If global bidder drop out from both licenses, then local bidders will again think that global bidder uses his equilibrium strategy and $p_1^* > v_{1A}, v_{1B}$. In other words, sequential rationality is always satisfied.

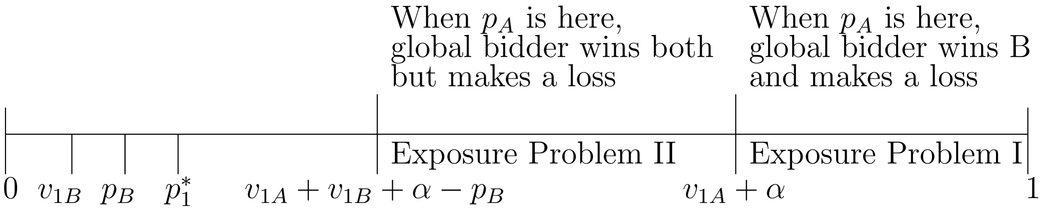

Figure 1: EXPOSURE PROBLEM

v_{1A} with v_{1B}), and there is one local bidder in each license.

$$p_1^* = \begin{cases} \frac{1}{2}\{v_{1B}+\alpha+1-(v_{1B}^2+1-2v_{1B}-\alpha^2+2v_{1B}\alpha+2\alpha-4v_{1A}\alpha)^{\frac{1}{2}}\}, \\ \qquad \textit{if } 0<v_{1A}<1-\alpha \textit{ and } 2(1-v_{1A})(v_{1A}-v_{1B})>\alpha^2; \\ \frac{1}{3}\{v_{1A}+v_{1B}+\alpha+1-((v_{1A}+v_{1B}+\alpha+1)^2-3(v_{1A}+\alpha)^2-6v_{1B})^{\frac{1}{2}}\}, \\ \qquad \textit{if } 0<v_{1A}<1-\alpha \textit{ and } 2(1-v_{1A})(v_{1A}-v_{1B})\leq\alpha^2; \\ \frac{1}{2}\{v_{1B}+\alpha+1-\{(v_{1B}+\alpha+1)^2-4(v_{1A}+v_{1B}+\alpha)+2+2v_{1A}^2\}^{\frac{1}{2}}\}, \\ \qquad \textit{if } 1-\alpha\leq v_{1A}<1 \textit{ and } 1+v_{1A}>2(v_{1B}+\alpha); \\ \frac{2(v_{1A}+v_{1B}+\alpha)-1}{3}, \\ \qquad \textit{if } 1-\alpha\leq v_{1A}<1 \textit{ and } 1+v_{1A}\leq 2(v_{1B}+\alpha). \end{cases} \tag{5}$$

The optimal drop-out price is a function that takes a unique value defined in the corollary above. For example, case $0 < v_{1A} < 1-\alpha$ and $2(1-v_{1A})(v_{1A}-v_{1B}) > \alpha^2$ implies that $p_1^* < v_{1A}$. If we assumed $v_{1A} = v_{1B}$, then our equilibrium drop out price in the corollary would coincide with the equilibrium drop out prices of Albano et. al (2001) and Goerre and Yuanchuan (2010). Hence, we generalize their results, and relax the identical valuations assumption.

2.1 Exposure Problem/Ex-post Loss

We now can discuss the exposure problem with the help of Figure 1. In the first type of exposure problem, the global bidder may win license B at a price above his stand alone valuation (i.e., $v_{1B} < p_B < p_1^*$) and lose the other license (i.e., $p_A > v_{1A}+\alpha$). This is the type of exposure problem Chakraborty (2004) focuses on. In the second type of exposure problem, the global bidder wins both licenses but incurs a loss. This is the case when he wins license B at $v_{1B} < p_B < p_1^*$ and wins license A at $v_{1A}+\alpha > p_A > v_{1A}+\alpha+v_{1B}-p_B$. Note that if he wins license A at the price $v_{1A}+\alpha+v_{1B}-p_B$, his payoff is zero. The global bidder stays in the auction for license A in order to minimize its loss from winning only

Table 1: **PROBABILITY OF EXPOSURE PROBLEM**

Synergy α	**Percentage of Exposure Problem 1** One Local Bidder	**Percentage of Exposure Problem 2** One Local Bidder	**Total Percentage** One Local Bidder	**Percentage of Inefficiency** One Local Bidder
Beta Distribution with parameters $\alpha = 1$ and $\beta = 4$				
0.2	2.85	0.64	3.48	7.84
0.4	2.29	2.43	4.72	5.45
0.6	0.54	1.59	2.13	2.15
0.8	0.04	0.62	0.66	0.66
Uniform Distribution				
0.2	1.17	0.40	1.57	3.29
0.4	1.13	1.07	2.20	4.22
0.6	0.69	1.51	2.20	4.01
0.8	0.36	1.25	1.61	2.62
Beta Distribution with parameters $\alpha = 4$ and $\beta = 1$.				
0.2	0.34	0.77	1.11	2.72
0.4	0.20	0.93	1.13	1.95
0.6	0.00	0.50	0.50	0.64
0.8	0.00	0.26	0.26	0.28

license B even if the price passes $v_{1A} + \alpha + v_{1B} - p_B$.

2.2 Simulations

In this subsection, through simulations, we determine the probability of the occurrence of ex-post loss for the global bidder and the inefficient allocation under various environments. As we noted, we say that exposure problem occurs when the global bidder wins one or both licenses with a loss ex-post. We have used MATLAB to write our simulation code. This code first draws the valuations for both the global and the local bidders from a given distribution function. We have one local bidder on each license and one global bidder on both licenses. One set of valuations correspond to one auction. We calculate the optimal drop-out price of the global bidder; local bidders' drop-out prices are their valuations. The global bidder's equilibrium drop-out price is updated as these local bidders drop out from the auction. If the global bidder does not win the first license, then no exposure problem occurs. If he wins the first license, then we calculate his updated price for the remaining license (unless the global bidder drops out from both licenses at the same time). We next determine whether the global bidder will win the remaining license at a positive profit (no exposure problem), at a loss (exposure problem II) or lose the remaining license (exposure problem I). Dividing the number of each of these events to the number of draws yields the probability of each event.[16]

We use three different distribution functions to draw valuations: uniform, beta distribution with $\alpha = 1$, and $\beta = 4$, and beta distribution with $\alpha = 4$ and $\beta = 1$. The second distribution is first order stochastically dominated by the uniform distribution, while the third one first order stochastically dominates the uniform distribution. We use one local bidder on each license.[17] We run simulations for four different synergy levels: 0.2, 0.4, 0.6, 0.8.[18] The 0.2 represents for small synergy, 0.4 and 0.6 represents middle synergy, and 0.8 represents a large synergy level. We report the results in TABLE 1.

We find that the global bidder may face exposure problem with probability 4.72 per cent, if the valuations are drawn from beta distribution with parameters $\alpha = 1$ and $\beta = 4$, and the synergy level is equal to 0.4. Note that the expected stand alone valuation for each license is 0.2 with this distribution. Hence, the global bidder's average valuation does not exceed 1 most of the time which result in more inefficient allocation and ex-post loss.

TABLE 1 also shows that the exposure problem occurs with the smallest probability among all these different distributions when the synergy level is 0.8. This is expected since the global bidder, after winning the first license, can bid very high for the remaining license (in most cases more than 1) but do not incur a loss due to high synergy level.

[16] In our simulation, to simplify calculations, we only consider cases in which the global bidder values the license A more than license B. After 10000 draws, we select only the valuations where the global bidder's valuation for license A is greater than license B. Hence, we are left with approximately 5000 draws. We used UNIX system of the University of Manitoba, and our laptops for the simulations. In the UNIX machine, it took more than four days to run each code.

[17] Given the complexity of the code we use, we feel that using one local bidder in our simulations are enough to draw reasonable conclusions though this can be extended to two local bidders in each license. Using three local bidders or more would extremely complicate the code since one has to keep track of updated prices every time a local bidder drops off.

[18] We also write codes for a finer synergy level of 0.1,0.2,...0.9 but we do not report them since no new insight is learned.

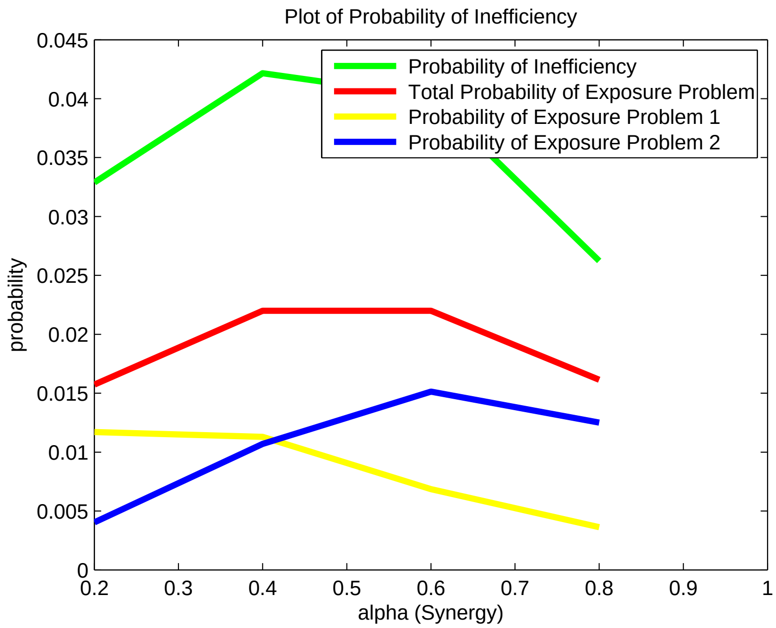

Figure 2: Valuations are drawn from Uniform Distribution.

In our simulations, Exposure problem I occurred most often when the synergy level is 0.4. In this case, the global bidder overbids to enjoy the middle level synergy, so he is very likely to make a substantial (potential) loss when he wins the first license. His optimal drop-out price for the remaining license is generally below 1; hence, the risk of losing the second license is high. Exposure problem II generally occurs the most when the synergy level is 0.6. After winning the first license, the global bidder will stay in the remaining license auction for a higher drop out price; hence, rather than exposure problem I, exposure problem II is likely to occur.

Of the two beta distribution, the one with $\beta = 1$ and the one with $\beta = 4$, we observe that the exposure problem occurs much less frequently with the first one. The reason is that with the former one, the valuations of the global bidder is likely to be higher; hence, even with a small synergy level, their optimal updated drop out price after winning one license is more than 1 in most cases. Therefore, it is less likely to have exposure problem. On the other hand, when the global bidder's updated optimal price is less than one, it is also more likely that the local bidder's valuation will be high so in such cases, one may see exposure problem. As the synergy increases, this case is less likely to happen.

In TABLE 1, we show that, for the Beta distribution with $\beta = 1$, the allocation is inefficient 7.84 per cent of the time when $\alpha = 0.2$. As α increases, global bidder wins both licenses more often without facing exposure problem that much, and this is the efficient outcome. Hence, we observe inefficient allocation only 0.66 per cent of the time for $\alpha = 0.8$. Most of the inefficiency is due to exposure problem.

We will talk about the role of inefficiency on revenue in the next section. Especially, when the inefficiency is due to exposure problem and when it is due to local bidders winning licenses inefficiently.

Before closing this section, we quickly talk about TABLE 2 which shows the optimal drop out prices for three different distributions. When the local bidders' valuation is expected to be low (when $\beta = 4$), optimal drop out price is higher compared to the cases where the local bidder's valuation is expected to be high (when $\beta = 1$).

3 Comparison with Vickrey Clarke Groves Auction

In this section, we will compare the revenue of our auction with the revenue of Vickrey Clarke Groves (VCG) auction.[19] VCG auction maximizes the expected payment of each agent among all mechanisms for allocating multiple objects that are efficient, incentive compatible, and individually rational.[20] In this auction, the seller will let the bidders bid on license A, license B and the whole package license A and B.

We, will first calculate the payment of each winner for all cases; that is, calculate the revenue of the seller. The payment of a winner (say player i) in this auction is the difference between the social welfare of the others if the bidder did not participate in the auction (denote this as $W^{-i}(x^{-i})$ where x^{-i} denote the bid of all players other than player i), and the welfare of the others when he participated in the auction, and bid truthfully (denote this as $W^{-i}(x)$, where x denote the bid of all players.) since truthful bidding is the weakly

[19] In the literature, there are many papers on VCG mechanism but not interested in exposure problem aspect such as Sano (2011) and Misra and PArkes (2009)

[20] See proposition 16.2 of Krishna (2010). In this section, we follow the notation of Krishna (2010) closely.

Table 2: **OPTIMAL DROP OUT PRICE**

Global Bidder's Valuation for License A v_{1A}	**Global Bidder's Valuation for License B** v_{1B}	**Uniform Distr.** p_1^U	**Beta Distr. with** $\alpha = 1$ **and** $\beta = 4$ $p_1^{\beta=4}$	**Beta Distr. with** $\alpha = 4$ **and** $\beta = 1$ $p_1^{\beta=1}$
		Synergy=0.2		
0.25	0.2	0.231	0.2796	0.2027
0.4	0.2	0.2641	0.2958	0.2086
0.6	0.2	0.3127	0.3225	0.2276
0.8	0.2	0.3683	0.36	0.289
0.81	0.4	0.5591	0.562	0.4951
		Synergy=0.4		
0.25	0.2	0.2909	0.4026	0.2215
0.4	0.2	0.3528	0.4671	0.2546
0.6	0.2	0.4536	0.52	0.3524
0.8	0.2	0.5551	0.56	0.489
0.81	0.4	0.7325	0.762	0.6951
		Synergy=0.6		
0.25	0.2	0.3787	0.5167	0.2865
0.4	0.2	0.4667	0.5912	0.3874
0.6	0.2	0.6	0.6886	0.5524
0.8	0.2	0.7268	0.76	0.689
0.81	0.4	0.8733	0.9044	0.8839
		Synergy=0.8		
0.25	0.2	0.5	0.6226	0.4451
0.4	0.2	0.6	0.6981	0.5807
0.6	0.2	0.7333	0.7987	0.7401
0.8	0.2	0.8667	0.8994	0.8775
0.81	0.4	1	1	1

Table 3: Bidders' Valuations.

A	B	AB
v_{1A}	v_{1B}	$v_{1A}+v_{1B}+\alpha$
v_{2A}	0	v_{2A}
0	v_{3B}	v_{3B}

dominant strategy.

The table below shows the valuations of the bidders. To give an example of how payments are calculated, let us assume that $v_{2A} + v_{1B} > v_{1A} + v_{1B} + \alpha > v_{2A} + v_{3B}$. VCG auction will allocate license A to local bidder A, and license B to global bidder. Payment of local bidder A is $(W^{-2}(x^{-2})) - W^{-2}(x) = (v_{1A} + v_{1B} + \alpha) - v_{1B} = v_{1A} + \alpha$. If local bidder A does not participate in the auction, then global bidder will win the package; hence, the welfare of the others $W^{-2}(x^{-2}) = v_{1A} + v_{1B} + \alpha$. When it participates in the auction, global bidder gets only license B, and local bidder B gets nothing; hence, the welfare of others in this case is $W^{-2}(x) = v_{1B}$.

Payment of the global bidder is: $(W^{-1}(x^{-1})) - W^{-1}(x) = (v_{2A} + v_{3B}) - v_{2A} = v_{3B}$. If the global bidder does not participate in the auction, local bidder A and B wins each license; hence, the welfare of others is the term inside the parenthesis. When the global bidder participates in the auction, local bidder A wins license A but the welfare of local bidder B is zero; hence the welfare of others in this case is just v_{2A}.

The total revenue of the seller in this case will be $v_{1A} + \alpha + v_{3B}$.

We summarize the revenue of the seller for all cases in proposition 9 in appendix.

We compare the revenue of the simultaneous ascending auction with those of the VCG auction through simulation methods. Our results are summarized in FIGURE 2. We run the simulations for $\alpha = 0.2, 0.4, 0.6, 0.8$ and three different distributions. All results show that VCG mechanism gives higher revenue especially when $\alpha = 0.2$. Therefore, unlike the complete information examples of VCG mechanism in the literature (e.g. Ausubel and Milgrom (2006)), we show that revenue is higher with VCG mechanism when licenses are

Table 4: Example of Revenue Comparison for Inefficient Allocations.

v_{1A}	v_{1B}	α	p_1^*	v_{2A}	v_{3B}	Revenue-Our Model	Revenue-VCG Mechanism
0.0963	0.0008	0.2	0.0718	0.1581	0.1247	$0.1681 = p_1^* + v_{1A}$	$0.2829 = v_{2A} + v_{3B}$
0.1149	0.1015	0.2	0.1677	0.31	0.16	$0.47 = v_{2A} + v_{3B}$	$0.3628 = 2(v_{1A} + v_{1B} + \alpha) - v_{2A} - v_{3B}$

not identical in an incomplete information setting.

When there is **inefficient allocation** in our auction, either global bidder wins (with exposure problem/ex-post loss), or local bidders win when global bidders would win in VCG auction. If most of the inefficient allocations are due to the first one, then our auction's revenue is expected to be higher. If most of the inefficient cases are due to the latter one, then VCG mechanism's revenue is expected to be higher. For example, when we look for Beta Distribution with $\beta = 4$ case, exposure problem is very high with 3.48 per cent for the synergy level of $\alpha = 0.2$; however, the overall inefficiency is 7.84 per cent. We can conclude that most inefficiency is due to local bidders winning licenses. As a result, VCG mechanism gives a higher revenue. When $\alpha = 0.6$ or $\alpha = 0.8$, we see that most inefficiency is due to exposure problem; hence, our auction gives a higher revenue compared to VCG mechanism. When $\alpha = 0.8$, inefficiencies disappear, and not surprisingly the revenue levels of both auctions converge.

To make our point, let us look at the sample valuations in TABLE 4. In the first row, our auction allocates goods inefficiently to two local bidders. With VCG, however, global bidder wins the auction, and pays a higher price. Hence, VCG auction gives a higher revenue. In the second row, global bidder wins both licenses with an ex-post loss. Hence, the revenue of our model is higher than VCG mechanism's revenue.

We prove that SAA auction increases revenue for cases in which global bidder wins licenses with ex-post loss, and VCG mechanism increases revenue for cases in which local bidders win the licenses but the allocation is inefficient.

Proposition 7 *When the global bidder wins licenses with an ex-post loss; that is, exposure problem occurs, then revenue of the VCG auction is lower than the simultaneous ascending auction (SAA); that is,* $R_{VCG} < R_{SAA}$.

Proposition 8 *When the local bidder wins the licenses and the allocation is inefficient, then revenue in VCG model is greater than SAA, that is,* $R_{VCG} > R_{SAA}$.

Unfortunately, one cannot say much when the allocation is efficient with SAA auction. There are cases in which SAA auction gives higher or lower revenue than the VCG auction in efficient allocations.

The literature writes that exposure problem lowers the revenue since the global bidder expects that it may end up with a loss; hence, does not over bid too much. Hence, local bidders win when the global bidder is supposed to win in an efficient auction like VCG. While this intuition is correct, there is another side of the story. The global bidder still bids above the stand alone valuation in a SAA auction; hence, it may end up winning licenses with a loss. This overbidding brings higher revenue compared in SAA auction compared to VCG auction.

4 More Than One Global Bidder

In this section, we will analyze the case where there is more than one global bidder. The difficulty in this case arises from the fact that each global bidder's optimal drop out price depends on the other global bidders' optimal drop out price which is a function of **private** valuations. As a result, each global bidder, while calculating its **own drop out price**, should at the same time calculate the **distribution of the other global bidder's drop out price**. This makes deriving an analytical result impossible.

To make our case, assume that there are two global bidders, Firm 1 and Firm 2, and one local bidder on each of license A and license B which are denoted as Firm 3 and Firm 4. Let the valuations v_{iA} for $i = 1, 2, 3$ and v_{jB} for $j = 1, 2, 4$ are drawn from the same distribution function $G(.)$. Everything else is the same as the previous section. In addition assume that $v_{iB} < v_{iA}$ for $i = 1, 2$.[21] Firm 1 will make a similar calculation as we discussed in one global bidder case –except that the price would depend on the drop out price of the second global

[21] This does not have to be true but our point is even if you assume such a restrictive assumption, you cannot calculate the optimal drop out prices.

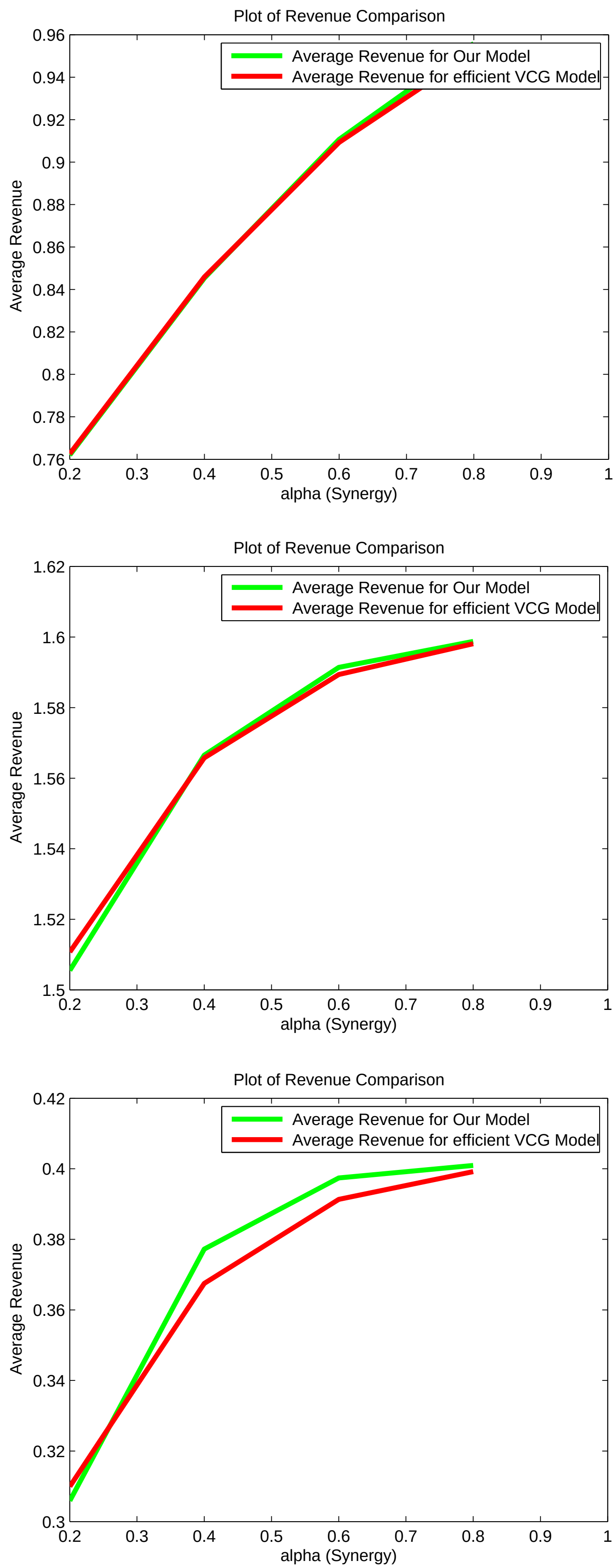

Figure 3:
Uniform Distribution (Top), Beta Distribution with $\alpha = 4$, $\beta = 1$, Beta Distribution with $\alpha = 1$, $\beta = 4$ (Bottom)

bidder which we denote as p_2^*– Specifically, he should make the following calculation while deciding to stay in the license B auction or not. This calculation is done based on a history in which none of the other bidders have dropped out yet.

$$E\Pi_1^1 = Max\{0, E\Big[([\int_p^{v_{1A}} (v_{1A} - Max\{v_{2A}, v_{3A}\}P(p_2^* < v_{4B}))dG(Max\{v_{2A}, v_{3A}\}|p) \quad (6)$$

$$+ \int_p^{v_{1A}} (v_{1A} - Max\{(v_{2A} + \alpha), v_{3A}\}P(p_2^* > v_{4B}))dG(Max\{(v_{2A} + \alpha), v_{3A}\}|p)])|p_2^*\Big]\} \quad (7)$$

$$E\Pi_1^2 = \int_p^{Min\{v_{1A}+\alpha,1\}} (V_1 - p - p_A)dG(Max\{v_{2A}, v_{3A}\}|p) + \int_{Min\{v_{1A}+\alpha,1\}}^1 (v_{1B} - p)dG(Max\{v_{2A}, v_{3A}\}|p) \quad (8)$$

The explanation of equation 5 is as follows. If this global bidder drops out from license B before the other global and local bidders, then it will continue on license A auction as a local bidder. Hence, it can only derive a benefit of v_{1A} if it wins the license. The price it will pay for license A depends on the result of whether the local or the other global bidder wins license B. If the local bidder B wins license B (this happens with probability $P(p_2^* < v_{4B})$ and we integrate this in the range p to v_{1A}), then the price of license A will be the maximum of local bidder A's valuation and the other global bidder's valuation for license which is $Max\{v_{2A}, v_{3A}\}$. Expectation is taken with respect to p_2^*.

Equation 7 analyze the case in which the other global bidder wins license B. This happens with probability $P(p_2^* > v_{4B})$ (in the range p to v_{1A}), then it can enjoy synergy and will bid until $v_{2A} + \alpha$ for license A. Then, the price global bidder 1 will pay is $Max\{(v_{2A} + \alpha), v_{3A}\}$.

Equation 8 analyzes the case in which Firm 1 wins license B, and then continue optimally on license A. This is the same as "one global bidder" case. The only difference is that the other global bidder is now a local bidder; hence, there is one more local bidder in the license A auction compared to the "one global case".

When we equate these integrals, we will find $p_1^*(v_{1A}, v_{1B}, \alpha, p_2^*(v_{2A}, v_{2B}, \alpha))$. Global bidder, Firm 1, does not know v_{2A} and v_{2B}, and hence, p_2^*. Since p_2^* is not known, Firm 1 must use its distribution which he does not even know the functional form. At the same time, the other global bidder, Firm 2, is making a similar calculation; however, he does not know $p_1^*(v_{1A}, v_{1B}, \alpha, p_2^*(v_{2A}, v_{2B}, \alpha))$. In other words, two global bidders will have two non-linear equations $p_1^*(v_{1A}, v_{1B}, \alpha, p_2^*(v_{2A}, v_{2B}, \alpha))$ and $p_2^*(v_{2A}, v_{2B}, \alpha, p_1^*(v_{1A}, v_{1B}, \alpha))$ that should be simultaneously solved, and to calculate these equations, they need to calculate the distribution of p_i^* for $i = 1, 2$. We are not sure whether this can be solved through simulations so it is left as an open problem.[22]

5 Conclusion and Discussion

We showed the optimal bidding strategies of global bidders when there are moderate synergies and the licenses are heterogeneous. We also analyzed exposure problem and its role on revenue extensively. Our revenue comparison of SAA and VCG auctions show that VCG mechanism gives a higher revenue when the synergy level is low.

We were able to show exposure problem can occur even when the global bidder wins all licenses. Literature has not studied heterogeneous license case with moderate synergies since it is technically challenging when one uses more than one global bidder. With this paper, we fill this gap. One of our contributions is to write a complicated code to calculate the probability of exposure problem. Our simulation results show that the exposure problem may be minor for some distributions but may be up to 4.3 per cent for some others.

Extending the results to n global bidders would be very complicated since the optimal strategies of global bidders (optimal drop out prices) should be determined jointly which in turn would depend on how many local and how many global bidders are still in the auction. Moreover, one has to know the distribution of the other global bidder's optimal drop out price while calculating the optimal drop out price! We leave this as an open problem, and, in this paper we follow the literature that use only one global bidder (e.g. Kagel and Levin (2005)).

[22]We suspect that other papers in the literature assumes identical licenses to avoid this problem In fact, Albano et. al (2006) writes "*In fact, for intermediate values of* $\alpha \in (0, 1)$ *and if* v_1 *and* v_2 *are different...;showing the existence of a PBE is already problematic in this case.*"

Our other contribution is comparing the revenue and the efficiency properties of the simultaneous ascending auction with those of the VCG auction. We show that when synergy level is small ($\alpha = 0.2$), the VCG mechanism generates more revenue. However, when valuations are such that exposure problem occurs, then SAA auction gives a higher revenue. When the local bidders win the license inefficiently (in SAA auction), the VCG mechanism gives a higher revenue.

References

[1] **Albano, Gian L., Fabrizio Germano, and Stefano Lovo.** 2001."A Comparison of Standard Multi-Unit Auctions with Synergies." *Economics Letters* 71: 55-60.

[2] **Albano, Gian L., Fabrizio Germano, and Stefano Lovo.** 2006. "Ascending Auctions for Multiple Objects: the Case for the Japanese Design." *Economic Theory* 28: 331-55.

[3] **Ausubel, Lawrence M., and Paul Milgrom.** 2006. "The Lovely But Lonely Vickrey Auction." Peter Cramton, Yoav Shoham, Richard Steinberg, eds. Combinatorial Auctions. MIT Press, Cambridge, MA.

[4] **Brusco, Sandro, and Giuseppe Lopomo.** 2002. "Collusion via Signaling in Simultaneous Ascending Bid Auctions with Heterogeneous Objects, with and without Complementarities." *The Review of Economic Studies* 69: 407-63.

[5] **Brusco, Sandro, and Giuseppe Lopomo.** 2009. "Simultaneous Ascending Auctions with Complementarities and Known Budget Constraints." *Economic Theory* 38: 105-24.

[6] **Chakraborty, Indranil.** 2004. "Multi-Unit Auctions with Synergy." *Economics Bulletin* 4: 1-14.

[7] **Chow, Yuenleng, and Abdullah Yavas.** 2009. "Auctions With Positive Synergies: Experimental Evidence." Manuscript.

[8] **Englmaier, Florian, Pablo Guillen, Loreto Llorente, Sander Onderstal, and Rupert Sausgruber.** 2009. "The Chopstick Auction: A Study of the Exposure Problem in Multi-Unit Auctions." *International Journal of Industrial Organization* 27: 286-91.

[9] **Goerre, Jacob K., and Lien Yuanchuan.** 2010. "An Equilibrium Analysis of the Simultaneous Ascending Auction." University of Zurich Working paper 428.

[10] **Harstad, Ronald M., and Michael H. Rothkopf.** 1995. "Withdrawable Bids as Winner's Curse Insurance." *Operations Research* 43: 983-94.

[11] **Kagel, John H., and Dan Levin.** 2005. "Multi-Unit Demand Auctions with Synergies: Behavior in Sealed-Bid versus Ascending-Bid Uniform-Price Auctions." *Games and Economic Behavior* 53: 170-207.

[12] **Katok, Elena, and Alvin E. Roth.** 2004. "Auctions of Homogeneous Goods with Increasing Returns: Experimental Comparison of Alternative Dutch Auctions." *Management Science* 50: 1044-63.

[13] **Krishna, Vijay, and Robert W. Rosenthal.** 1996. "Simultaneous Auctions with Synergies." *Games and Economic Behavior* 17: 1-31.

[14] **Porter, David P.** 1999. "The Effect of Bid Withdrawal in a Multi-Object Auction." *Review of Economic Design* 4: 73-97.

[15] **Menicucci, Domenico.** 2003. "Optimal Two-Object Auctions with Synergies." *Review of Economic Design* 8: 143-64.

[16] **Mishra, Debasis, and David C. Parkes.** 2009. "Multi-Item Vickrey-Dutch Auctions." *Games and Economic Behavior* 66: 326-47.

[17] **Rosenthal, Robert W., and Ruqu Wang.** 1996. "Simultaneous Auctions with Synergies and Common Values." *Games and Economic Behavior* 17: 32-55.

[18] **Sano, Ryuji.** 2011. “Non-bidding equilibrium in an ascending core-selecting auction.” *Games and Economic Behavior*, Forthcoming.

[19] **Wang, Ruqu.** 2000. “Bidding and Renegotiation in Procurement Auctions.” *European Economic Review* 44: 1577-97.

[20] **Zheng, Charles Z.** 2008. “Jump Bidding and Overconcentration in Decentralized Simultaneous Ascending Auctions.” Iowa State University Department of Economics Staff General Research Papers 12698.

A Mathematical Appendix

Proof of Lemma 4

We will prove that there is a unique optimal drop out price by solving $E\Pi_1^1 = E\Pi_1^2$. Here, we assume that $v_{1B} < v_{1A}$.[23] We have four cases.

Case I: In this case, we will assume $v_{1A}+\alpha < 1$ and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A+(v_{1B}-v_{1A}) < 0$. We will show that this implies $p_1^* < v_{1A}$ (which in turn implies $E\Pi_1^1 > 0$).

First, we show that there exists a unique solution that makes equations 1 and 2 equal, and this is the optimal drop out price p_1^*. We define a new function, $J(p,m) = E\Pi_1^1 - E\Pi_1^2$. To prove uniqueness, we will show that this function is monotonically increasing and it is negative when $p = v_{1B}$ (by lemma 2, p cannot be less than v_{1B}) and is positive when $p = v_{1A}$. Hence, there must be a unique root at the interval $v_{1B} < p < v_{1A}$.

$J(p,m) = \int_p^{v_{1A}}(v_{1A}-p_A)g(p_A|p)dp_A - \int_p^{v_{1A}+\alpha}(V_1-p-p_A)g(p_A|p)dp_A$
$- \int_{v_{1A}+\alpha}^1(v_{1B}-p)g(p_A|p)dp_A$.

By using $\int_p^1 g(p_A|p)dp_A = 1$ since $g(p_A|p)$ is a probability density function on the support $[p,1]$, we have $(v_{1B}-p)\int_p^1 g(p_A|p)dp_A = v_{1B}-p$, we can re-write it as
$\int_p^{v_{1A}}(v_{1A}-p_A)g(p_A|p)dp_A - \int_p^{v_{1A}+\alpha}(v_{1A}+\alpha-p_A)g(p_A|p)dp_A - (v_{1B}-p)$
By using integration by parts twice $\int udv = uv - \int vdu$, first we assume that $u = v_{1A}-p_A$ and $v = G(p_A|p)$; then assume that $u = v_{1A}+\alpha-p_A$ and $v = G(p_A|p)$, thus, $dv = g(p_A|p)dp_A$,

[23] The other case, $v_{1B} > v_{1A}$, is symmetric.

we have

$= (v_{1A} - p_A)G(p_A|p) \mid_p^{v_{1A}} - \int_p^{v_{1A}} G(p_A|p)d(v_{1A} - p_A)$

$- (v_{1A} + \alpha - p_A)G(p_A|p) \mid_p^{v_{1A}+\alpha} + \int_p^{v_{1A}+\alpha} G(p_A|p)d(v_{1A} + \alpha - p_A) - (v_{1B} - p)$

$= \int_p^{v_{1A}} G(p_A|p)dp_A - \int_p^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p) = -\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p)$

We take partial derivative of $J(p,m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = \frac{\partial}{\partial p}[-\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)] + 1 > 0$

It is positive since the term $\frac{\partial}{\partial p}[\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)]$ is negative. As the lower limit of the integral increases, the value of the expression decreases (does not increase) if the term inside is non-negative which is true since it is a cumulative distribution function. We must also show that $\frac{\partial G(p_A|p)}{\partial p} \leq 0$ to prove this. While one can easily see that this is correct (as p increases the cumulative distribution conditional on p decreases), we will give a formal proof by using Leibniz's rule when necessary.

$\Leftrightarrow \frac{\partial G(p_A|p)}{\partial p} = \frac{\partial[(\frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv})^{m-1}]}{\partial p}$

$= -(m-1)f(p)\frac{(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}} + (m-1)f(p)\frac{(\int_p^{p_A} f(v)dv)^{m-1}}{(\int_p^1 f(v)dv)^{m}}$

$= \frac{(m-1)f(p)(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}}[-1 + \frac{\int_p^{p_A} f(v)dv}{\int_p^1 f(v)dv}]$

$= \frac{(m-1)f(p)(\int_p^{p_A} f(v)dv)^{m-2}}{(\int_p^1 f(v)dv)^{m-1}}[-1 + F(p_A|p)] < 0$ (≤ 0 only if $p_A = 1$).

Thus, $J(p,m)$ is monotonically increasing function of p, when $v_{1B} \leq p < v_{1A}$.

If $p = v_{1B}$, then $J(v_{1B}) = \int_{v_{1B}}^{v_{1A}} G(p_A|\alpha)dp_A - \int_{v_{1B}}^{v_{1A}+\alpha} G(p_A|v_{1B})dp_A$

$= -\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|v_{1B})dp_A < 0$. (note that we prove this for the case $v_{1A} > v_{1B}$)

If $p = v_{1A}$, $J(v_{1A}) = 0 - \int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - v_{1A}) > 0$, then our assumption $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A + (v_{1B} - v_{1A}) < 0$ implies that $J(p = v_{1A}) > 0$.

Hence, there is a unique root, p_1^* in the such that $v_{1B} < p_1^* < v_{1A}$.

Next, we show that as the number of active firms in license A auction decreases, the optimal drop out price will increase. We will use the implicit function theorem for this:

$\Leftrightarrow \frac{dp_1^*}{dm} = -\frac{\frac{\partial J(p_1^*,m)}{\partial m}}{\frac{\partial J(p_1^*,m)}{\partial p_1^*}} < 0.$

We have already shown that $\frac{\partial J(p_1^*,m)}{\partial p_1^*} > 0$.

Since $J(p,m) = \int_p^{v_{1A}} G(p_A|p)dp_A - \int_p^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p)$.

Since the lower limits of the two integrals are the same, we can re-write this as $J(p,m) = -\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p)$.

By using Leibniz's rule for differentiation under the integral sign, we take partial derivative of $J(p,m)$ with respect to m and have,

$\frac{\partial J(p,m)}{\partial m} = -\int_{v_{1A}}^{v_{1A}+\alpha} \frac{\partial G(p_A|p)}{\partial m} dp_A = -\int_{v_{1A}}^{v_{1A}+\alpha} \ln(F(p_A|p))G(p_A|p)dp_A > 0$, since $\frac{\partial G(p_A|p)}{\partial m} = \ln(F(p_A|p))G(p_A|p) < 0$. Hence, we show that $\frac{\partial J(p|m)}{\partial m} > 0$ holds.

By the implicit function theorem, we show that the optimal drop out price increases as the number of local firms, m, decreases.

Since $\frac{\partial J(p,m)}{\partial p} > 0$ and $\frac{\partial J(p,m)}{\partial m} > 0$, we have, $\frac{dp_1^*}{dm} = -\frac{\frac{\partial F(p_1^*,m)}{\partial m}}{\frac{\partial F(p_1^*,m)}{\partial p_1^*}} < 0$

Case II: In this case, we will assume that $v_{1A} + \alpha < 1$ and $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|v_{1A})dp_A + (v_{1B} - v_{1A}) \geq 0$ which we will show that this implies $p_1^* \geq v_{1A}$. And this condition in turn implies that $E\Pi_1^1 = 0$, since the global bidder drops from both licenses.

Now let $J(p,m) = E\Pi_1^1 - E\Pi_1^2$

$J(p,m) = 0 - \int_p^{v_{1A}+\alpha}(V_1 - p - p_A)g(p_A|p)dp_A - \int_{v_{1A}+\alpha}^1 (v_{1B} - p)g(p_A|p)dp_A$
$= -\int_p^{v_{1A}+\alpha} G(p_A|p)dp_A - (v_{1B} - p)$.

When $p \geq v_{1A}$, we take partial derivative of $J(p,m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = -\frac{\partial}{\partial p}[\int_p^{v_{1A}+\alpha} G(p_A|p)dp_A] + 1 > 0$, since $\frac{\partial G(p_A|p)}{\partial p} < 0$.

Thus, $J(p,m)$ is monotonically increasing function of p, when $v_{1A} \leq p \leq v_{1A} + \alpha$.

Our assumption $\int_{v_{1A}}^{v_{1A}+\alpha} G(p_A|v_{1A})dp_A + (v_{1B} - v_{1A}) \geq 0$ implies that $J(p = v_{1A}, m) \leq 0$. If $p = v_{1A} + \alpha$, then $J(v_{1A} + \alpha) = 0 - 0 - (v_{1B} - v_{1A} + \alpha) > 0$. Thus, there is a unique solution, p_1^*, in the interval $[v_{1A}, v_{1A} + \alpha)$.

Next, we show that when the number of active firms in license A auction decreases, this optimal drop out price will increase.

We take partial derivative of $J(p,m)$ with respect to m, we have,

$\frac{\partial J(p,m)}{\partial m} = -\int_p^{v_{1A}+\alpha} \ln(F(p_A|p))G(p_A|p)dp_A > 0$.

Since $\frac{\partial J(p,m)}{\partial p} > 0$ and $\frac{\partial J(p,m)}{\partial m} > 0$, we have,

$\frac{dp_1^*}{dm} = -\frac{\frac{\partial J(p_1^*,m)}{\partial m}}{\frac{\partial J(p_1^*,m)}{\partial p_1^*}} < 0$

Case III: In this case, we will assume that $v_{1A} + \alpha > 1$ and $\int_{v_{1A}}^{1} G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) < 0$ which implies $v_{1A} < p_1^*$. And this condition in turn implies that $E\Pi_1^1 > 0$.

Now let $J(p, m) = E\Pi_1^1 - E\Pi_1^2$

$J(p, m) = \int_p^{v_{1A}} (v_{1A} - p_A)g(p_A|p)dp_A - \int_p^1 (v_{1A} + v_{1B} + \alpha - p - p_A)g(p_A|p)dp_A$

Again, we assume that $u = v_{1A} - p_A$ and $v = G(p_A|p)$ for the first integral; then assume that $u = v_{1A}+v_{1B}+\alpha-p-p_A$ and $v = G(p_A|p)$ for the second integral, thus, $dv = g(p_A|p)dp_A$, By using integration by parts, we have

$= (v_{1A} - p_A)G(p_A|p) \mid_p^{v_{1A}} + \int_p^{v_{1A}} G(p_A|p)dp_A$

$- (v_{1A} + v_{1B} + \alpha - p - p_A)G(p_A|p) \mid_p^1 - \int_p^1 G(p_A|p)dp_A$

$= \int_p^{v_{1A}} G(p_A|p)dp_A - (v_{1A} + v_{1B} + \alpha - p - 1) - \int_p^1 G(p_A|p)dp_A$

$= -(v_{1A} + v_{1B} + \alpha - p - 1) - \int_{v_{1A}}^1 G(p_A|p)dp_A$

We take partial derivative of $J(p, m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = \frac{\partial}{\partial p}[-\int_{v_{1A}}^1 G(p_A|p)] + 1 > 0$

It is positive since the term $\frac{\partial}{\partial p}[\int_{v_{1A}}^1 G(p_A|p)]$ is negative. And we have shown that $\frac{\partial G(p_A|p)}{\partial p} \leq 0$. Thus, $J(p, m)$ is monotonically increasing function of p, when $v_{1B} \leq p < v_{1A}$.

If $p = v_{1B}$, then $J(v_{1B}) = -\int_{v_{1A}}^1 G(p_A|v_{1B})dp_A - (v_{1A} + \alpha - 1) < 0$ since $v_{1A} + \alpha > 1$.

If $p = v_{1A}$, $J(v_{1A}) = 0 - \int_{v_{1A}}^1 G(p_A|v_{1A})dp_A - (v_{1B} + \alpha - 1) > 0$ since $v_{1B} < v_{1A}$ and $v_{1A} + \alpha > 1$, then our assumption $\int_{v_{1A}}^1 G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) < 0$ implies that $J(p = v_{1A}) > 0$.

Hence, there is a unique root in the interval $v_{1B} < p < v_{1A}$.

We skip to show that as the number of active firms in license A auction decreases, the optimal drop out price will increase, since we have done this in Case I.

Case IV: In this case, we will assume that $v_{1A} + \alpha > 1$ and $\int_{v_{1A}}^1 G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) \geq 0$ which implies $p_1^* \geq v_{1A}$. And this condition in turn implies that $E\Pi_1^1 = 0$.

Now let $J(p, m) = E\Pi_1^1 - E\Pi_1^2$

$J(p, m) = 0 - \int_p^1 G(p_A|p)dp_A - (v_{1A} + v_{1B} + \alpha - p - 1)$.

When $p > v_{1A}$, we take partial derivative of $J(p, m)$ with respect to p, we have,

$\frac{\partial J(p,m)}{\partial p} = -\frac{\partial}{\partial p}[\int_p^1 G(p_A|p)dp_A] + 1 > 0$, since $\frac{\partial G(p_A|p)}{\partial p} < 0$.

Thus, $J(p,m)$ is monotonically increasing function of p, when $v_{1A} \le p \le 1$.

Our assumption $\int_{v_{1A}}^{1} G(p_A|v_{1A})dp_A + (v_{1B} + \alpha - 1) \ge 0$ implies that $J(p = v_{1A}) \le 0$. If $p = 1$, then $J(1) = 0 - 0 - (v_{1B} + v_{1A} + \alpha - 2) > 0$. Since $v_{1B} + v_{1A} + \alpha < 2$ by our average valuation is less than 1 assumption in the lemma. Thus, there is a unique solution, p_1^*, in the interval $[v_{1A}, 1)$.

We also skip to show that when the number of active firms in license A auction decreases, this optimal drop out price will increase which has been proven in Case II.

Proposition 9 *Suppose that there is one global bidder and one local bidder bidding for each license. In the VCG auction, the seller's revenue will be as follows depending on the valuations of the bidders.*[24]

CASE I: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha < v_{2A} + v_{3B}$.

A) (Local bidders win each license) And suppose that $v_{1A} < v_{2A}$ *and* $v_{1B} < v_{3B}$. *There are four sub cases to consider.*

i) $v_{3B} > v_{1B} + \alpha$ *and* $v_{2A} > v_{1A} + \alpha$, *then the revenue is* $v_{1A} + v_{1B}$.

ii) $v_{3B} < v_{1B} + \alpha$ *and* $v_{2A} < v_{1A} + \alpha$, *then the revenue is* $2(v_{1A} + v_{1B} + \alpha) - v_{3B} - v_{2A}$.

iii) $v_{3B} > v_{1B} + \alpha$ *and* $v_{2A} < v_{1A} + \alpha$, *then the revenue is* $2v_{1A} + v_{1B} + \alpha - v_{2A}$.

iv) $v_{3B} < v_{1B} + \alpha$ *and* $v_{2A} > v_{1A} + \alpha$, *then the revenue is* $v_{1A} + 2v_{1B} + \alpha - v_{3B}$.

B) (Local bidder wins A, and global bidder wins B) And suppose that $v_{2A} > v_{1A}$ *and* $v_{3B} < v_{1B}$. *Then, the revenue is* $v_{1A} + \alpha + v_{3B}$.

C) (Local bidder wins B, and global bidder wins A) And suppose that $v_{2A} < v_{1A}$ *and* $v_{3B} > v_{1A}$. *Then, the revenue is* $v_{1B} + \alpha + v_{2A}$

CASE II: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha > v_{2A} + v_{3B}$.

A) (Global bidder wins both licenses) And suppose that $v_{1A} < v_{2A}$ *and* $v_{1B} < v_{3B}$. *Then, the revenue is* $v_{2A} + v_{3B}$.

B) (Global bidder wins both licenses) And suppose that $v_{1A} > v_{2A}$ *and* $v_{1B} > v_{3B}$. *Then, the revenue is* $v_{2A} + v_{3B}$.

[24] This result can easily be extended to one global and many local bidders case. Since, we use one global bidder and one local bidder in each license in the simulations, to save the notation, we give the result for a special case.

C) (Global bidder wins license A, local bidder B wins license B) And suppose that $v_{2A} < v_{1A}$ *and* $v_{3B} > v_{1B} + \alpha$*. Then, the revenue is* $v_{2A} + v_{1B} + \alpha$*.*

D) (Global bidder wins license B, local bidder A wins license A) And suppose that $v_{2A} > v_{1A} + \alpha$ *and* $v_{3B} < v_{1B}$*. Then, the revenue is* $v_{3B} + v_{1A} + \alpha$*.*

Proof of Proposition 9 Suppose that there is one global bidder and one local bidder bidding for each license. In the VCG auction, the seller's revenue will be as follows depending on the valuations of the bidders.

CASE I: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha < v_{2A} + v_{3B}$.

A) (Local bidders win each license) And suppose that $v_{1A} < v_{2A}$ and $v_{1B} < v_{3B}$. There are four sub cases to consider.

i) $v_{3B} > v_{1B} + \alpha$ and $v_{2A} > v_{1A} + \alpha$, then the revenue is $v_{1A} + v_{1B}$.

since local bidder A's payment would be $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{3B} - (0 + v_{3B}) = v_{1A}$ and local bidder B's payment would be $W(x^{-3}) - W^{-3}(x) = v_{1B} + v_{2A} - v_{2A} = v_{1B}$

ii) $v_{3B} < v_{1B} + \alpha$ and $v_{2A} < v_{1A} + \alpha$, then the revenue is $2(v_{1A} + v_{1B} + \alpha) - v_{3B} - v_{2A}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{3B}) = v_{1A} + v_{1B} + \alpha - v_{3B}$ and $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{2A}) = v_{1A} + v_{1B} + \alpha - v_{2A}$

iii) $v_{3B} > v_{1B} + \alpha$ and $v_{2A} < v_{1A} + \alpha$, then the revenue is $2v_{1A} + v_{1B} + \alpha - v_{2A}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{3B} - (0 + v_{3B}) = v_{1A}$, and $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{2A}) = v_{1A} + v_{1B} + \alpha - v_{2A}$

iv) $v_{3B} < v_{1B} + \alpha$ and $v_{2A} > v_{1A} + \alpha$, then the revenue is $v_{1A} + 2v_{1B} + \alpha - v_{3B}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{3B}) = v_{1A} + v_{1B} + \alpha - v_{3B}$ and $W(x^{-3}) - W^{-3}(x) = v_{1B} + v_{2A} - v_{2A} = v_{1B}$

B) (Local bidder wins A, and global bidder wins B) And suppose that $v_{2A} > v_{1A}$ and $v_{3B} < v_{1B}$. Then, the revenue is $v_{1A} + \alpha + v_{3B}$.

since $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{1B}) = v_{1A} + \alpha$, and $W(x^{-1}) - W^{-1}(x) = v_{3B} + v_{2A} - v_{2A} = v_{3B}$

C) (Local bidder wins B, and global bidder wins A) And suppose that $v_{2A} < v_{1A}$ and $v_{3B} > v_{1B}$. Then, the revenue is $v_{1B} + \alpha + v_{2A}$

since $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (0 + v_{1A}) = v_{1B} + \alpha$, and $W(x^{-1}) - W^{-1}(x) = v_{3B} + v_{2A} - v_{3B} = v_{2A}$

CASE II: Suppose that the valuations are such that $v_{1A} + v_{1B} + \alpha > v_{2A} + v_{3B}$.

A) (Global bidder wins both licenses) And suppose that $v_{1A} < v_{2A}$ and $v_{1B} < v_{3B}$. Then, the revenue is $v_{2A} + v_{3B}$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (0 + 0) = v_{2A} + v_{3B}$.

B) (Global bidder wins both licenses) And suppose that $v_{1A} > v_{2A}$ and $v_{1B} > v_{3B}$. Then, the revenue is $v_{2A} + v_{3B}$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (0 + 0) = v_{2A} + v_{3B}$.

C) (Global bidder wins license A, local bidder B wins license B) And suppose that $v_{2A} < v_{1A}$ and $v_{3B} > v_{1B} + \alpha$. Then, the revenue is $v_{2A} + v_{1B} + \alpha$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (v_{3B} + 0) = v_{2A}$ and $W(x^{-3}) - W^{-3}(x) = v_{1A} + v_{1B} + \alpha - (v_{1A} + 0) = v_{1B} + \alpha$.

D) (Global bidder wins license B, local bidder A wins license A) And suppose that $v_{2A} > v_{1A} + \alpha$ and $v_{3B} < v_{1B}$. Then, the revenue is $v_{3B} + v_{1A} + \alpha$.

since $W(x^{-1}) - W^{-1}(x) = v_{2A} + v_{3B} - (v_{2A} + 0) = v_{3B}$ and $W(x^{-2}) - W^{-2}(x) = v_{1A} + v_{1B} + \alpha - (v_{1B} + 0) = v_{1A} + \alpha$.

Proof of Proposition 7 When there is an exposure problem, then revenue in VCG model, R_{VCG}, is lower than revenue in SAA, R_{SAA}.

First, we consider when there is type 1 exposure problem, there are cases:

Case 1: when $v_{1B} < v_{3B} < p_1^*$ (global bidder wins license B with a loss), and $v_{2A} > v_{1A} + \alpha$ (but loses license A), then $R_{SAA} = v_{1A} + \alpha + v_{3B}$.

For the above parameter space, there are two possible subcases in VCG model: one is when $v_{3B} > v_{1B} + \alpha$, and $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$ (the latter condition makes this an inefficient allocation), then $R_{VCG} = v_{1A} + v_{1B}$ which is the Case I: A)i) from Proposition 9. Since $v_{3B} > v_{1B}$, we have $R_{VCG} < R_{SAA}$.

The other one is when $v_{3B} < v_{1B} + \alpha$, and $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$, then $R_{VCG} = v_{1A} + v_{1B} + \alpha + v_{1B} - v_{3B}$ which is the Case I: A)iv) from Proposition 9. Since $v_{3B} > v_{1B}$, we have $R_{VCG} < R_{SAA}$.

Case 2: This is the symmetric case when $v_{2A} > v_{1A}$, $v_{2A} < p_1^*$(global bidder wins license A with a loss), and $v_{3B} > v_{1B} + \alpha$ (but loses license B), then $R_{SAA} = v_{1B} + \alpha + v_{2A}$.

For the above parameter space, there are two possible subcases in VCG model: one is when $v_{3B} > v_{1B}$, $v_{2A} > v_{1A}$, $v_{2A} > v_{1A}+\alpha$, $v_{3B} > v_{1B}+\alpha$, and $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$(the latter condition makes this an inefficient allocation), then $R_{VCG} = v_{1A} + v_{1B}$ which is the Case I: A)i) from Proposition 9. Since $v_{2A} > v_{1A}$, we have $R_{VCG} < R_{SAA}$.

The other one is when $v_{3B} > v_{1B}$, $v_{2A} > v_{1A}$, $v_{2A} < v_{1A} + \alpha$, $v_{3B} > v_{1B} + \alpha$, and $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$, then $R_{VCG} = v_{1A}+v_{1B}+\alpha+v_{1A}-v_{2A}$ which is the Case I: A)iii) from Proposition 9. Since $v_{2A} > v_{1A}$, we have $R_{VCG} < R_{SAA}$.

Then, we consider when there is type 2 exposure problem, there are two kinds of cases.

Case 1: When $v_{3B} > v_{1B}$, $v_{2A} < v_{1A} + \alpha$, and $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$(global bidder wins license A and B with a loss), then $R_{SAA} = v_{2A} + v_{3B}$.

For the above parameter space, there are three possible subcases in VCG model: The first one is when $v_{3B} > v_{1B}$, $v_{2A} > v_{1A}$, $v_{2A} < v_{1A}+\alpha$, $v_{3B} > v_{1B}+\alpha$, and $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$, then $R_{VCG} = 2(v_{1A} + v_{1B} + \alpha) - v_{2A} - v_{3B}$ which is the Case I: A)ii) from Proposition 9. Since $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$, we have $R_{VCG} < R_{SAA}$.

The second one is when $v_{3B} > v_{1B}$, $v_{2A} > v_{1A}$, $v_{2A} < v_{1A} + \alpha$, $v_{3B} < v_{1B} + \alpha$, and $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$, then $R_{VCG} = v_{1A}+v_{1B}+\alpha+v_{1A}-v_{2A}$ which is the Case I: A)iii) from Proposition 9. Since $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$ and $v_{2A} > v_{1A}$, we have $R_{VCG} < R_{SAA}$.

The third one is when $v_{3B} > v_{1B}$, $v_{2A} < v_{1A}$, and $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$, then $R_{VCG} = v_{1B} + \alpha + v_{2A}$ which is the Case I: C) from Proposition 9. Since $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$ and $v_{2A} < v_{1A}$, we have, $v_{3B} > v_{1B} + \alpha$. So we have $R_{VCG} < R_{SAA}$.

Case 2: This is the symmetric of case 1 above. When $v_{2A} > v_{1A}$, $v_{3B} < v_{1B} + \alpha$, $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$(global bidder wins license A and B with a loss), then $R_{SAA} = v_{2A} + v_{3B}$.

For the above parameter space, there are three possible cases in VCG model: The first one is when $v_{2A} > v_{1A}$, $v_{3B} > v_{1B}$, $v_{3B} < v_{1B}+\alpha$, $v_{2A} < v_{1A}+\alpha$, and $v_{2A}+v_{3B} > v_{1A}+v_{1B}+\alpha$, then $R_{VCG} = 2(v_{1A} + v_{1B} + \alpha) - v_{2A} - v_{3B}$ which is the Case I: A)ii) from Proposition 9. Since $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$, we have $R_{VCG} < R_{SAA}$.

The second one is when $v_{2A} > v_{1A}$, $v_{3B} > v_{1B}$, $v_{3B} < v_{1B} + \alpha$, $v_{2A} > v_{1A} + \alpha$, and $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$, then $R_{VCG} = v_{1A} + v_{1B} + \alpha + v_{1B} - v_{3B}$ which is the Case I: A)iv) from Proposition 9. Since $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$ and $v_{3B} > v_{1B}$, we have $R_{VCG} < R_{SAA}$.

The third one is when $v_{3B} < v_{1B}$, $v_{2A} > v_{1A}$, $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$, then $R_{VCG} = v_{1A} + \alpha + v_{3B}$which is the Case I: B) from Proposition 9. Since $v_{2A} + v_{3B} > v_{1A} + v_{1B} + \alpha$ and $v_{3B} < v_{1B}$, we have, $v_{2A} > v_{1A} + \alpha$. So we have $R_{VCG} < R_{SAA}$.

Proof of Proposition 8 When the allocation is inefficient (except for two types of exposure problem) in SAA, then revenue in VCG model is greater than that in SAA, that is, $R_{VCG} > R_{SAA}$.

There are two cases.

Case 1: Global bidder loses both licenses in SAA model, but wins both in VCG model. When $v_{3B} > p_1^* > v_{1B}$(global bidder loses licese B), $v_{2A} > v_{1A}$(global bidder loses license A), and $v_{2A} + v_{3B} < v_{1A} + v_{1B} + \alpha$(this condition makes an inefficient allocation), if $v_{1A} > p_1^*$, then $R_{SAA} = p_1^* + v_{1A}$ or if $v_{1A} < p_1^*$, then $R_{SAA} = p_1^* + p_1^*$.

There is one corresponding possible subcase in VCG model which is the Case I: A)iv) from Proposition 9. when $v_{3B} > v_{1B}$, $v_{2A} > v_{1A}$, and $v_{2A} + v_{3B} < v_{1A} + v_{1B} + \alpha$, then $R_{VCG} = v_{2A} + v_{3B}$. When $v_{1A} < p_1^*$, since $v_{3B} > p_1^*$,and $v_{2A} > v_{1A}$, we have $R_{VCG} < R_{SAA}$; or when $v_{1A} > p_1^*$, since $v_{3B} > p_1^*$,and $v_{2A} > v_{1A}$, we also have $R_{VCG} > R_{SAA}$.

Case 2: Global bidder wins license A and local bidder wins license B in SAA model.[25] When $v_{3B} > p_1^*$(global bidder loses licese B), $v_{2A} < v_{1A}$(global bidder wins licese A), $v_{2A} > p_1^*$, and $v_{1A} + v_{3B} < v_{1A} + v_{1B} + \alpha$, then $R_{SAA} = p_1^* + \alpha + v_{2A}$.

In such case, there is one possible subcase in VCG model which is the Case II: B)iv) from Proposition 9 when $v_{3B} < v_{1B}+\alpha$, and $v_{2A} < v_{1A}$, then we can derive $v_{2A}+v_{3B} < v_{1A}+v_{1B}+\alpha$, then $R_{VCG} = v_{3B} + v_{2A}$. Since $v_{3B} > p_1^*$, then we have $R_{VCG} > R_{SAA}$.

[25]Since we assume $v_{1A} > v_{1B}$, we do not have the case global bidder wins license B and local bidder license A. If we remove the assumption, the proof can be written symmetrically.