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Endogenous Competition Alters the Structure of Optimal Auctions

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Abstract

Potential bidders respond to a seller's choice of auction mechanism for a common-value or affiliated-values asset by endogenous decisions whether to incur an information-acquisition cost (and observe a private estimate), or forgo competing. Privately informed participants decide whether to incur a bid-preparation cost and pay an entry fee, or cease competing. Auction rules and information flows are quite general; participation decisions may be simultaneous or sequential. The resulting revenue identity for any auction mechanism implies that optimal auctions are allocatively efficient; a nontrivial reserve price is revenue-inferior. Optimal auctions are otherwise contentless: any auction that sells without reserve becomes optimal by adjusting any one of the continuous, spanning parameters, e.g., the entry fee. Seller's surplus-extracting tools are now substitutes, not complements. Many econometric studies of auction markets are seen to be flawed in their identification of the number of bidders.

D44; D82; C72; Keywords: optimal auctions, endegenous bidder participation, affiliated-values, common-value auctions, surplus-extracting devices

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1 Introduction

How should an owner or auctioneer select a selling procedure when bidders' value estimates for an asset are private information? That fundamental question has for centuries received a variety of answers from experienced auctioneers, who in different markets persist in conducting their business in quite different ways. In contrast, theoretical models of "optimal auctions" with rational risk-neutral bidders have tended to provide a unique answer.[1]

While the particular answer provided depends fragilely on the model assumed, optimal auctions in the literature share two common characteristics. First, the optimal auction is inefficient (unless surplus can be fully extracted), primarily due to a nontrivial *reserve price*.[2] Second, it is a complicated mechanism. Depending on the particular assumptions, it has involved: distribution- and bidder-specific reserve prices, disjoint sets of prices at which seller refuses to sell, requiring payments from losing bidders that vary with their bids and rivals', requiring bidders to accept lotteries with unboundedly large losses, or to accept lotteries before their terms are specified.[3]

When the assumption of a fixed number of bidders (often simply treated as notation) is realistically opened to having the number of bidders respond endogenously to the expected profitability of bidding,[4] the strikingly different characterizations that arise are shown here also to be strikingly robust. The optimal auction is efficient, and the argument for a nontrivial reserve price absent. The characterization of equilibrium expected revenue becomes more general, and indeed can be expressed as a function solely of the expected number of privately informed participants.

This paper's second major purpose, less well known but no less important: to demonstrate

[1] In essence, the models cited in the following footnote each define a very narrow equivalence class of auctions, and show that optimal auctions all fall in a single equivalence class, which serves to characterize nearly all auction forms as necessarily suboptimal, even with adjustments in parameters of that auction form.

[2] Myerson [1981], Harris and Raviv [1981] and Riley and Samuelson [1981] derive optimal auctions when bidders' private information (their types) are independent. Of these models, Myerson's is most general. All revolve around a nontrivial reseve price (below which the seller prevents the asset from ever being sold); so do more recent papers (see surveys in Klemperer [2000] and Krishna [2002]). The only optimal auctions attaining efficiency are in models that use strong informational assumptions and correlated types to extract full surplus: Crémer and McLean [1985], [1988], McAfee, McMillan and Reny [1989], McAfee and Reny [1992] and Crémer, Spiegel and Zheng [2009]. The criticism of these models in Robert [1991], that the weakest form of limited liability or infinitesimal risk aversion renders them discontinuously suboptimal, is similar in spirit to the present effort. Mares and Harstad [2007] provide an accessible treatment of necessary and sufficient conditions for full surplus extraction.

[3] Examples of these complications, in order: Harris and Raviv [1981], Myerson [1981], Crémer and McLean [1985], McAfee, McMillan and Reny [1989], McAfee and Reny [1992].

[4] Headline-grabbing auctions–airwaves licenses, privatization of governmental enterprises, offshore oil leases, investment-quality art, initial public offerings, acquisitions of new, established, and distressed corporations–all fit the mold of endogenous decisons about whether to compete, rather than exogenous number of bidders. So do such mundane markets as used-car auctions, timber sales and routine art auctions. (Buying at auction and selling at retail is a sensible stylization of the art-gallery business.)

that when bidder participation is modeled as a rational decision, characterization of an auction as optimal does precious little to constrain the structure of the auction. Different auctioneers can employ different mechanisms without necessarily implying that any are behaving suboptimally. The Content Theorem below yields a most useful interpretation, given that all auction theory models abstract from reality: an unmodeled aspect of a particular market–prior practice, the need for the speed of a Dutch auction, unwarranted costliness of congregating bidders in time or space, bidders' preference for dynamic ascending prices, or for a mechanism protecting bidders' private information from a bidtaker or rival or observer who lacks knowledge of the bidder's beliefs or of the distribution of private information but can observe his bids–each can be accommodated without sacrifice of efficiency or revenue. Indeed, adjusting any spanning variable suffices.

These characterizations stem from two overriding assumptions: symmetry and single-dimensional types (though quite strong assumptions, these are ubiquitous in auction theory), and two sensible constraints on auction forms: anonymity and uniqueness. If potential bidders are asymmetric in their behavior, in their beliefs or in the distribution generating their private information (types), multiple equilibria are unavoidable, and participation decisions either cannot be analyzed or cannot be separated from an ad hoc equilibrium selection (which will bury the interesting economics). The same problem arises if auction rules are not anonymous, and arises unless the seller is constrained to auction rules which admit a unique symmetric equilibrium. Very little is known about auction theory when a bidder's private information cannot be summarized by a single random variable; the impossibility theorem for equilibrium existence in Jackson [2009] suggests broad, fundamental problems.

With these two assumptions and two constraints, a model quite general by the usual standards of auction theory can be analyzed. Affiliated-values auctions are considered, an extension of the "General Symmetric Model" in Milgrom and Weber [1982] to allow for an endogenous number of bidders. Thus, both private-values and common-value motivations can be present. N "potential bidders", not yet privately informed, make decisions whether to obtain costly private information; the decisions may be simultaneous or sequential (the sequential model is treated in Appendix B). The (endogenous) n "participants" then acquire private information (an estimate of asset value, equivalently, their type, in the usual terminology of games of incomplete information) and may–or may not–learn the realization n of the number of privately informed participants who are competing. Next a (possibly costly) decision as to whether to prepare for and conduct bidding is made; the a "actual bidders" who do so may–or may not–learn the value of a before bids are submitted.

Seller's information disclosure decisions include options as to how accurately to disclose, when to disclose, and whether to disclose publicly or privately.[5] It adds both generality and realism to incorporate resource costs both before and after participants become privately informed; limiting revenue generation to arise *after* participants become privately informed avoids incorporating what is in essence a lump-sum tax.

Hence, it is *determination of the level of competition by expected profitability decisions* that alters the structure of optimal auctions, *not details* of how this determination is modeled.

I emphasize at the outset that the model avoids any assumption of monotone equilibrium, in particular avoiding the assumption of a screening level (a threshold level of private information above which a participant chooses to pay the entry fee). An impossibility theorem by Landsberger and Tsirelson [2000] makes this complication critical.[6] This aspect, together with allowing for possible sequential participation decisions (Appendix B), for general affiliated valuations, for resource costs facing potential bidders both *before* (an information-acquisition cost) and *after* becoming privately informed (a bid-preparation cost), for seller to wield the widest variety of surplus-extracting tools, as well as allowing for players either to observe or not observe the number of players still competing at each stage, greatly distinguish the generality of this model from prior auction models which endogenize the number of bidders.[7]

2 A Small Example

Auctions are where market transactions usefully reflect private information because the mechanism can be explicit about the rules underlying transaction determination. This advantage of auctions forces robust models to deal with a variety of potentially cumbersome rule choices. So I begin with a simple illustration of the difference endogenous participation makes.

The example asset has a common value $T \sim U[0,10]$. First assume each of an exogenously

[5] The model treats situations where a seller offers an asset to potential bidders who decide whether to compete to buy. A corresponding model where a buyer details a contractual obligation, and potential bidders decide whether to compete to supply, has completely corresponding results.

[6] A more accessible discussion of this theroem is in Landsberger [2007].

[7] Harstad [1990] introduces the notion that the number of bidders ought to be considered an endogenous variable, in a simpler model employing monotone equilibrium and with a smaller set of surplus-extracting tools. Levin and Smith [1994] also depend on monotone equilibrium, and critically on assumptions that the seller [i] cannot disclose an appraisal, and [ii] cannot impose an entry fee after bidders have private information, for their result supporting a positive reserve price. Chakraborty and Kosmopoulou [2001] partially specify a simpler model employing monotone equilibrium and with a smaller set of surplus-extracting tools, and argue for a zero entry fee when negative entry fees are precluded. McAfee [1993] describes a steady state in a private-values model where an exogenous number of bidders is endogenously partitioned across sellers; the relation to his paper is reconsidered below in footnote 33.

specified 3 bidders observes an estimate $X_j \sim U[t, t+1|T=t]$. Suppose seller held auction form m_2, a second-price auction with no reserve price, no entry fee and no information about asset value disclosed by seller. In m_2's symmetric equilibrium, expected revenue would be 4.833, the winner's expected profit would be 0.167, and an individual bidder's *ex ante* expected profit one-third of that. If seller switched to auction form m_E, an otherwise identical auction conducted under English auction rules, and still drew 3 bidders, expected revenue would rise to 4.875, the winner's expected profit falling to 0.125.

Now adjust, instead to have $N = 5$ potential bidders decide whether to compete. If a potential bidder wishes to acquire a privately-observed estimate of asset value, he must incur an information-acquisition cost $c = 0.177$; each participant j who does so draws his estimate from the same distribution as in the prior paragraph, $X_j \sim U[t, t+1|T=t]$. For simplicity, this example sets the bid-preparation cost to 0, and assumes that participants learn the number of participants in the auction before selecting bidding strategies. Calculations are simplified for both second-price and English auctions by a pathology of the double-uniform distribution: for a participant observing an estimate $X_j \in [0.5, 10.5]$, expected profitability of continuing to compete is constant (it is lower on $[10.5, 11]$ and 0 on $[0, 0.5]$).

Suppose each of the 5 potential bidders participates (that is, pays c to acquire an estimate, necessary to compete) with probability $\pi = 0.6$. Then the probability μ_a of a bids submitted is listed in the middle row of Table 1, with the expected number of bids submitted, 3, shown as $\overline{a}$ in the right-hand column. However, any given potential bidder, if he chooses to compete, instead calculates the probabilities of $n-1$ rival participants when 4 rival potential bidders each participate with probability 0.6; this yields as expected number of bids submitted the λ_a calculations in the bottom row. Thus, he rationally expects 3.4 bidders on average.

Table 1

# bids:	0	1	2	3	4	5	$\overline{a}$
μ_a	0.01	0.08	0.23	0.35	0.21	0.05	3.0
λ_a	0	0.03	0.15	0.35	0.35	0.13	3.4

With endogenous participation, the auction rules need further specification; assume both m_2 and m_E inform participants of the number n of participants before bidding strategies are selected. Then for the second-price auction m_2, if a potential bidder's 4 rivals all take part with probability 0.6 (and each then submits the equilibrium bid for the estimate he observed), his expected profit should he compete will be 0.177, making him just indifferent over whether he competes. The resulting equilibrium attains expected revenue 4.42 (including allowance for 0 revenue in the 0.11-probability [rounded] event that less than 2 bids are submitted).

Should the same 5 potential bidders participate with the same probability $\pi = 0.6$ in the English auction m_E, expected revenue would be 4.46. However, this is not an equilibrium: assuming a potential bidder's 4 rivals all take part with probability 0.6, his expected profit should he compete will be less, only 0.166, so he would prefer not to pay the 0.177 information-acquisition cost. Rather, in equilibrium for auction m_E, the 5 potential bidders participate with probability 0.59, yielding the 0.177 expected profit that makes each indifferent over competing. The equilibrium revenue for m_E exceeds that for m_2 by less than 0.002.

As potential bidders are in equilibrium indifferent over participating, Theorem 2 below shows that equilibrium expected revenue equals the expected excess of gains from trade over resource costs incurred (here, those costs simplify to $c\pi\overline{a}$); Corollary 3 shows that revenue can be expressed solely as a function of the expected number of bidders. For these parameters, any auction that attains 3.29 bidders on average (equivalently, a participation probability of approximately 0.574) will reach the maximum attainable level of revenue, 4.4224. One way of attaining this is to keep the second-price auction m_2, but impose an entry fee of $\varphi = 0.038$ (which is just enough to force down the equilibrium participation probability to the desired level). Among many other ways to reach maximum attainable revenue is to use the English auction m_E, but impose an entry fee of 0.028; the desired entry fee is lower because an English auciton with an exogenous n bidders reduces the winner's expected profit relative to a second-price auction. If some unmodeled aspect of the marketplace yielded a reason to conduct a first-price auction, there would be some entry fee, larger than the 0.038 for the second-price auction, that would allow a first-price auction to be optimal.

3 A General Model with Endogenous Participation

Begin with the notion that potential bidders choose among a variety of auctions, and other uncertain economic opportunities, in which to invest their attention, time and money. Only a segment of the extensive form of such a game, that relating to a particular auction, appears explicitly here. One indivisible asset is sold in the explicit model. A subset of the (exogenously determined) N potential bidders will participate, and a subset of the n participants will become the a actual bidders. It will be convenient to describe seller's strategies both in terms of recognized aspects of auctions, such as pricing rules and information disclosure policies, and also as elements of an abstract set.

The game segment unfolds as follows, cf. Figure 1. First, seller announces an *auction mechanism* $M := (m, \varphi, r) \in \mathbb{M} \subset (\mathcal{M} \times \Re \times \Re_+) \subset \Re^{d+1}$, where m is an *auction form*, $\mathcal{M}$ the set of auction forms, φ an *entry fee*, and r a *reserve price*. An auction form m specifies not just pricing rules, but the entire flow of information and hence the nature of the extensive form continuation. For example, $m = m_0$ might specify a second-price auction with seller releasing an uncensored independent appraisal to all participants, as well as specifying that neither participants nor active bidders learn their number before bidding. Or $m = m_1$ might specify an English (oral ascending) auction without any seller-released public information, but with an appraisal privately revealed to one actual bidder chosen at random, in which the number of participants is not learned but the number of actual bidders is, and alternating recognition rules determine the probabilistic revelation of bidders' exit prices to remaining bidders. A particular seller in a particular situation may face additional constraints: he may, for example, find credibly imposing a nontrivial reserve price impossible, or may be unable to inform participants of the number of competitors who acquired private information, or may not have a reputation that would allow using a second-price auction without bidders assuming he could well insert a fake bid just below the highest bid;[8] all such constraints are treated via making $\mathbb{M}$ the feasible set of auction mechanisms for a particular auction. The set $\mathbb{M}$ consists of discrete choices in some variables (e.g., English or second-price or first-price auction) and continuous choices in others (e.g., the signal-to-noise ratio of a public announcement of an appraisal possessed by seller); without loss of generality, it can be embedded in a real space. For later convenience, the dimensionality of this space is $d+1$. Note that when a seller has the option of credibly announcing how many participants are still competing (or how many actual bidders) before continuing, or of preventing the participants (or actual bidders) from knowing n (or a), seller's choice is simply modeled as a choice between two (otherwise identical) auction forms, just as if it were a choice between first- and second-price auction rules.

Second, a pool of potential bidders $\mathbb{N} := \{1, \ldots, N\}$ simultaneously select probabilities π_i of becoming a *participant* in this auction, basing those decisions on M (or select these probabilities sequentially, considered in Appendix B).[9] Participation has two consequences: each participant j obtains some private information $X_j \in \mathcal{X}$ about the asset's value to him (call this j's *estimate*), and

[8] The impact of privately revealed information is considered in Mares and Harstad [2003]; alternating recognition rules for English auctions are analyzed in Harstad and Rothkopf [2000]. Impacts of bidtaker cheating are considered in Rothkopf and Harstad [1995].

[9] Similar results to those in the main text when participation decisions are sequential depend on an equilibrium selection favorable to seller.

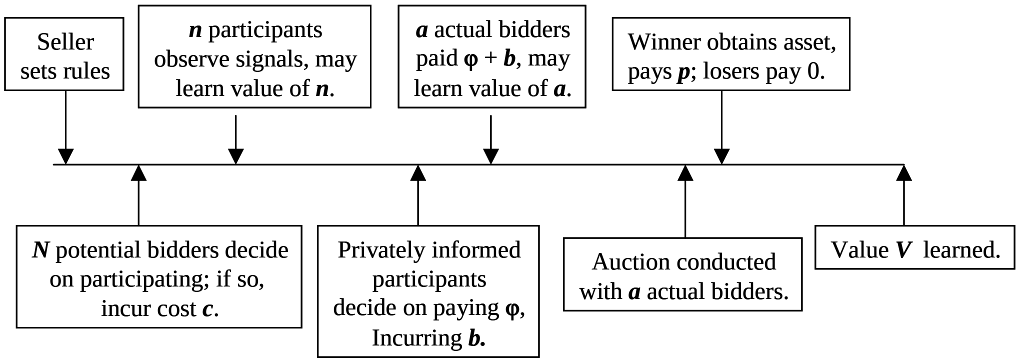

Figure 1: Time Line

each incurs an information-acquisition cost, $c > 0$. The information-acquisition cost is exogenously specified, and does not generate revenue for seller. It may represent resource costs of acquiring information about the quality of the asset being auctioned and/or represent foregone profitable opportunities (e.g., passing on the potentially profitable option to participate in another auction occurring elsewhere).[10] These two possibilities alter the interpretation of results, an issue considered in Concluding Remarks. This cost is likely to vary across auctions, but c is the same for all potential bidders in a given auction, and invariant to the mechanism by which the auction is run. The payoff of a potential bidder who does not participate is normalized to 0.

Third, each participant $j = 1, \ldots, n$ decides whether or not to incur a *bid-preparation cost* $b \geq 0$ (which does not accrue to seller),[11] plus pay the entry fee φ to seller and thereby become an *actual bidder*, based on information available at the time. This information includes the auction mechanism M, and participant j's private estimate X_j. If the component m of M characterizing the auction form specifies that participants are informed of the number n of participants, then n is taken into account. If n is not known, then the vector $(\pi_1, \ldots, \pi_N)$ of rational participation probabilities of potential bidders is taken into account. A participant who chooses not to continue attains a payoff of $-c$.

Fourth, each actual bidder $k = 1, \ldots, a$ selects a bidding strategy for the auction form m with reserve price r. In addition to M and X_k, n if known, and $(\pi_1, \ldots, \pi_N)$ if n is not known, this decision takes into account the number a of actual bidders if the auction form m releases this

[10] This consideration is missed if a view of substitute auctions is not at least implicitly present. Unwillingness of an additional potential bidder to participate need not imply zero (gross) expected profit.

[11] The bid-preparation cost is treated as the same no matter what auction mechanism is employed. This assumption is not innocuous; I return to it in Concluding Remarks.

information. If not, the bidding decision takes into account the functional structure of participants' decisions on whether to pay the entry fee, and includes strategizing to learn about a (and perhaps useful inferences about rivals' private information) as soon as information flows permit.

The winning bidder pays a price p for the asset, if this price is no less than the reserve price r; otherwise the asset goes unsold, which implies that all actual bidders would then be losers.[12] Losing actual bidders attain a payoff of $-\varphi - b - c$.

Asset value to a particular participant observing estimate X_i a continuous function $v(T, X_i)$, increasing in both variables (common across participants, in that v does not have a subscript). Here T is a common trend to asset values, or perhaps the common-value component of asset value. A sale yields the winning bidder a payoff of $v(T, X_i) - p - \varphi - b - c$. Seller's payoff is

$$u^S = \begin{cases} 0, & \text{if } a = 0 \text{ or } p < r, \\ p + a\varphi, & \text{if } a \geq 1 \text{ and } p \geq r. \end{cases}$$ [13]

3.1 Assumptions: Auction Environment

$\mathcal{A}$.1. The infinite sequence $\{X_1, X_2, \ldots\}$ from which participants will observe estimates is a sequence of exchangeable, positively affiliated, real-valued random variables with nonatomic measure η, and marginal η_1 onto support $\mathcal{X} \subset \Re$.

Affiliation is defined and characterized in Milgrom and Weber [1982], pp. 1098-1100 and 1118-1121; it is referred to as the MLRP (monotone likelihood ratio property) in several auction models. Roughly, affiliation means that higher realizations for any subset of the variables $\{X_1, X_2, \ldots\}$ make higher realizations for any disjoint subset more likely. Exchangeability means that the joint distribution is unaffected by any finite permutation of the indices. Let $\mathbf{X}_z = \{X_1, \ldots, X_z\}$, and $T_z = \left(\frac{1}{z}\right) \sum_{i=1}^{z} X_i$.

$\mathcal{A}$.2. The common trend $T = \lim_{z \to \infty} T_z$; $c + b < \inf_\eta v(T, X) < \sup_\eta v(T, X) < \infty$.

A variant of DiFinetti's Theorem justifies the use of a limit in $\mathcal{A}$.2:

[12] Notation that is already more cumbersome than might be hoped for would find significant additional complication if the seller were allowed to announce a vector of reserve prices r_a, with the r_a that corresponded to the number a of bids actually submitted enforced after bids were submitted. That complication would not affect any of the results below. In particular, the result in Levin and Smith [1994] that a nontrivial reserve price would be employed at least in the case where only 1 actual bidder showed up is still seen to depend on their assumptions that the seller cannot utilize an entry fee, and cannot publicly reveal such information as an appraisal.

[13] Considering the value to seller of an unsold asset to be 0 is, as usual, a harmless normalization. It bears emphasis, however, that failing to meet the reserve price implies that the seller is irrevocably constrained from ever offering this asset to this set of potential bidders in the future (this assumption is nearly ubiquitous in auction theory, though seldom mentioned). I return to this consideration in Concluding Remarks.

Theorem 1 *(Kingman [1980]) Given $\mathcal{A}$.1, the sequence $\{T_1, T_2, \ldots\}$ almost surely converges pointwise. Moreover, conditional on T, the $\{X_i\}$ are mutually independent.*

Letting the common-value component equal the asymptotic mean is without loss of generality (Milgrom and Weber [1986]).[14]

3.2 Assumptions: Auction Rules

$\mathcal{A}$.3. The price paid is an anonymous, nondecreasing, continuous function of the profile of actual bids submitted.[15]

$\mathcal{A}$.4. Each auction form $m \in \mathbb{M}$ provides all bidders with the same strategy set, determines a winning bidder anonymously,[16] and attains a unique symmetric equilibrium continuation for any exogenously specified binomial distribution of the number of actual bidders (including degenerate).

Allowing for payments to or from losing bidders greatly complicates the notation, but would not change any results below. Uniqueness of the symmetric equilibrium continuation (a constraint on the set of mechanisms available to seller) is critical to being able to predict the profitability of participating and actually bidding; it is satisfied for a wide variety of auction forms.[17]

3.3 Assumptions: Behavior

$\mathcal{A}$.5. All $N \geq 2$ potential bidders are risk-neutral.[18]

$\mathcal{A}$.6. Symmetric behavior: each potential bidder selects the same probability π of participating, each participant selects the same function of known and inferred information to determine whether to actually bid, and each actual bidder selects the same bid function. These selections constitute a Bayesian equilibrium continuation.

[14] I chose these assumptions about informational and valuational variables to be as natural as possible for the purpose of modeling endogenous participation. They are also mild in comparison to the bulk of auction theory models. However, the results below are quite general, and do not depend on the specific nature of $\mathcal{A}$.1 and $\mathcal{A}$.2. For example, all results extend, under virtually unchanged proofs, if the corresponding assumptions in Pesendorfer and Swinkels [1997] are substituted.

[15] The sort of revenue-maximizing, non-capricious discriminaton across bidders in Myerson [1981] has already been ruled out by exchangeability (in $\mathcal{A}$.1). The sort of capricious discriminaton contemplated in McAfee, McMillan and Reny [1989] is ruled out here, solely for notational ease. Footnote 29 below explains how their mechanism, which extracts full surplus whenever the exogenous number of bidders is at least two, becomes revenue-inferior with endogenous bidder participation.

[16] Though it may be natural, nothing depends on this being the highest bidder. For example, the lottery-qualification auction (Harstad and Bordley [1986]) meets these assumptions.

[17] Cf. Levin and Harstad [1986], Bikhchandani and Riley [1991], Pesendorfer and Swinkels [1997], Harstad and Rothkopf [2000], and Maskin and Riley [2000].

[18] All results below readily extend to the case where all N potential bidders have the same concave utility function, with much more cumbersome notation and no further insight.

If, in addition, seller selects the mechanism M to maximize expected revenue given the assumed behavior of bidders, a full Bayesian equilibrium is attained. As our focus is on the behavior that various announcements of M will induce, and thus upon the expected revenue attained, equilibrium continuation is the key assumption.[19]

Anonymity appears in $\mathcal{A}.3$ and $\mathcal{A}.4$, distributional symmetry of private information in $\mathcal{A}.1$, and symmetry of behavior in $\mathcal{A}.6$ (symmetry of beliefs is suppressed, as it would add notation without altering content). The last part of $\mathcal{A}.1$, that $\mathcal{X} \subset \Re$, imposes single-dimensionality of private information. Seller is constrained in $\mathcal{A}.4$ to announce auction rules satisfying uniqueness.[20]

4 The Participation Decision

In this model, the equilibrium expected number of participants is not invariant to seller's choice of auction mechanism. Rather, it adjusts to the auction mechanism M so that expected profit equals information-acquisition cost. The straightforward logic is, ultimately, independent of many details of the mechanism.

In the symmetric equilibrium, each of the N potential bidders will participate with the same probability $\pi(M)$; the expected number of participants is then $\overline{n}(M) = N\pi(M)$. This section exploits symmetry to consider the participation decision of potential bidder 1 when $N-1$ rival potential bidders all participate with probability π; the main text merely summarizes the intuition, with the horrendously cumbersome details diverted to Appendix A (and the case of sequential participation decisions to Appendix B). Appendix A develops the set of estimates bidder 1 could observe that would imply a sufficient expected profitability of competing to justify paying the entry fee. This development differs somewhat according to whether or not the n participants learn the number of participants; both cases are complicated by the need to avoid assuming a threshold determines such a set. The set characterized is used to identify, in Conclusion 1, the ex ante probabilities that [i] a particular potential bidder, if he chooses to become a participant, will draw a signal justifying continued competition, [ii] he will face any particular amount of bidding competition, and [iii] only the first α participants will become actual bidders (which is different

[19] Note that there exist Nash equilibria in which seller selects an otherwise inferior M' because, for example, all N potential bidders respond to any $M \neq M'$ by not participating, or otherwise punishing seller. These equilibria fail to be subgame-perfect; in ignoring them, I follow a standard but usually implicit practice.

[20] Which auction forms meet that constraint may depend on particular details of parameters; for example, only for narrow classes of η has a unique equilibrium been found when seller's information is privately disclosed to a randomly selected bidder (Mares and Harstad [2003]).

from [ii]).Probability [iii] is denoted $\mu_a(M, \pi, n)$, though degenerate in π if participants learn their number, and in n if they don't. Next, it develops the probability s_r that the reserve price r is met. Finally, using a function $p(\cdot)$ that expresses the expected auction price conditioned on various arguments, expected profitability is characterized, first if the bidding stage is reached, then moving backward to the decision to become an actual bidder and then to the decision to participate. The "expected profit equals information-acquisition cost" characterization is Conclusion 2.

For all the gory details, the underlying intuition models the only possibilities: a privately informed participant continues competing to become an actual bidder if and only if the expected profitability of doing so exceeds the costs not yet sunk (entry fee φ and bid-preparation cost b); a not-yet-privately-informed potential bidder competes (participates) if and only if the expected profitability of doing so (net of φ and b, should they be incurred) at least covers the information-acquisition cost c. In equilibrium, expected profitability exactly equals c.

Appendix A details definitions for the following notation:

Table 2

$V(M, \pi, n, \varphi, t)$	Expected asset value for mechanism M, participation probability π,
	n participants, entry fee φ, common trend t (ex ante, before learning X_j),
$p(M, a, \pi, n, t)$	Expected price, given M, a actual bidders, π, n, t,
$s(M, a, \pi, n)$	(Abbreviated s_r), sale probability given M, which specifies the reserve price r, for a, π, n,
$\mu_a(M, \pi, n)$	Probability that only the first a participants will become actual bidders, for M, π, n,
$\beta(z, Z, \zeta)$	Binomial probability of z successes in Z trials, given success probability ζ.

Depending on the information flows specified by a mechanism M, each of the first four functions above will typically be degenerate in one of its variables.[21]

5 General Revenue Formulation

Begin, naturally, at the end: with a specification of seller's expected revenue, conditional on assuming $a \geq 1$ actual bidders and $n \geq a$ participants:

$$s_r E_\eta [p(M, a, \pi, n, \cdot)] + a\varphi,$$

[21] For example, $p(\cdot)$ is degenerate in a for a second-price auction, μ_a is degenerate in π if the auction form informs bidders of n, and degenerate in n if bidders are not so informed.

which simply sums the price paid by the winner (multiplied by the probability of a sale), and entry fees paid by all actual bidders. It is harmless to condition on the common-value component's expected value, and simplifies the characterization to come.

Stepping back by replacing an assumed number of actual bidders and then of participants with equilibrium continuation probabilities gives our specification of expected revenue:

$$\mathcal{R}(M,n) = \sum_a \left(s_r E_\eta \left\{E_\eta \left[p\left(M,a,\pi,n,\cdot\right)|T\right]\right\} + a\varphi\right)\binom{n}{a}\mu_a\left(M,\pi,n\right). \tag{1}$$

$$\begin{aligned} R(M) &= \sum_n \mathcal{R}(M,n)\,\beta\left[n,N,\pi(M)\right] \\ &= \sum_n \left\{\sum_a \left(s_r E_\eta \left\{E_\eta \left[p(\cdot)|T\right]\right\} + a\varphi\right)\binom{n}{a}\mu_a\left(M,\pi,n\right)\right\}\beta_n. \end{aligned} \tag{2}$$

Equation (2) is still a simple sum of the price paid and entry fees, itself summed over the objective probabilities of a actual bidders and n participants (there are $\binom{n}{a}$ ways that $\mu_a(M,\pi,n)$ might correctly predict the number of actual bidders). (The summation harmlessly ignores the events of 0 participants and 0 actual bidders, which contribute 0 revenue.) To interpret expected revenue, natural definitions of the expected value transferred and expected number of actual bidders, are invoked:

$$\begin{aligned} \overline{V}(M) &= \sum_n \sum_a s_r E_\eta \left\{V\left[M,\pi(M),n,\varphi,T\right]\right\}\binom{n}{a}\mu_a\left(M,\pi,n\right)\beta_n, \\ \overline{a}(M) &= \sum_n \left\{\sum_a a\binom{n}{a}\mu_a\left(M,\pi,n\right)\right\}\beta_n. \end{aligned}$$

(These summations also harmlessly ignore the cases $n = 0$, $a = 0$.) Note that these definitions depend on the mechanism; in particular, $\overline{V}(M)$ treats as a zero transfer an asset that does not sell.

Theorem 2 *(The Fundamental Revenue Identity): In symmetric equilibrium continuation with endogenous bidder participation, for any $M \in \mathbb{M}$,*

$$R(M) = \overline{V}(M) - b\overline{a}(M) - c\overline{n}(M). \tag{3}$$

Theorem 2 is proven simply by separating out terms in (2) that are zero by equilibrium participation (eq. (13) in Appendix A). The notationally cumbersome details are diverted to Appendix C.

In simple language, the Identity says that revenue in symmetric equilibrium continuation is equal in expectation to the expected value transferred (or, if you wish, expected gains from trade) less aggregate participation and bid-preparation costs.[22] It is particularly important that this identity provides a simple formula for revenue for all M; there is no need for separate formulas for first-price, second-price, and English auctions, or for different information-revealing policies (except to determine $\pi[M]$), and the entry fee does not directly enter the calculation. The reserve price enters only through the probability of a sale.

Viewing efficiency as the sum of expected surplus of seller and all N potential bidders, Theorem 2 yields a general and striking contrast to prior optimal auctions models (in which revenue is maximized by enforcing allocative inefficiencies):

Corollary 1 *The Bayesian equilibrium in which seller maximizes expected revenue is allocatively efficient. Indeed, seller's preferences over any set of auction mechanisms match those of an efficient social planner.*

Proof. The right-hand-side of (3) is an efficiency measure, and in equilibrium continuation is also seller's objective. ■

Due to mixed-strategy participation decisions, when $\pi(M) < 1$, there is a probability $(1-\pi)^N$ of the event that the asset goes unsold because all N potential bidders happen not to participate. This is not an inefficiency: as it can only be avoided by making the auction so attractive that all N potential bidders participate, and the added resource costs accrue to seller, seller and a central planner identically accept this event. A more subtle issue: since the set of estimates justifying continued competition are not in general upper contours, in a revenue-maximal auction, there can be a positive probability that the participant observing the highest estimate chooses to cease competing rather than pay the entry fee, with the asset then sold to a participant who values it less

[22] No result approaching comparable generality is in the literature, but this Theorem has many antecedents. Relative to Levin and Smith [1994], for example, it is original in extension to affiliated-values environments, in its allowance for costs incurred both before and after bidders observe private information, in allowance for numbers of participants and actual bidders to be either learned or inferred, in the number and variety of surplus-extracting devices allowed for, and in dealing with the impossibility of a screening level.

In special cases, a corresponding result is found by Samuelson [1985] and Hausch and Li [1990], can be calculated in the example of Theorem 5.2 in Milgrom [1981], and found as an asymptotic approximation in Matthews [1984] (where the number of bidders is not necessarily an equilibrium level, but the information-acquisition costs are). Theorem 2 verifies shortcuts taken, but not justified explicitly, in equations (2) and (3) in Harstad [1990]. French and McCormick [1984] discuss a similar heuristic feature of first-price, common-value auctions, but do not provide a complete model or equilibrium characterization. McAfee and McMillan [1987b] assert the corresponding equation for a nonstochastic but supposedly endogenous n, without justification either for the equation or the source of n, and proceed incorrectly to dismiss the possibility that seller could enhance expected revenue via a positive entry fee.

highly than he does. While this reduces the expected value transferred, seller and a social planner both accept it. A sufficient reduction in φ will eliminate the possibility of selling to an inefficient acquirer, but the lower φ will increase $\overline{n}$, and with it increase the resource costs borne by seller.[23]

6 Inferiority of a Nontrivial Reserve Price

A seller can attain the entire interval of equilibrium participation probabilities, 0 through 1; this result is shown next for a second-price auction, chosen purely for convenience. The range of equilibrium values of π is attained by varying only the entry fee φ (including possibly $\varphi < 0$, reimbursing a fraction of participation and bid-preparation costs), while keeping the reserve price fixed at $r = 0$. Let $\overline{M}_\varphi = (\overline{m}, \varphi, 0)$, where $\overline{m}$ is a "vanilla" second-price auction with no disclosure of seller's information, and with n and a revealed to bidders; $\overline{M}_\varphi$ "sells without reserve."

Theorem 3 *For any $\pi_0 \in [0, 1]$, there exists an entry fee φ_0 such that $\pi\left(\overline{M}_{\varphi_0}\right) = \pi_0$.*

The proof (in Appendix C) simply sets up the Intermediate Value Theorem.

A reserve price r is *nontrivial* if at least one actual bidder does not guarantee a sale, that is, if there is an $a > 0$ such that $\{s(M, a, \pi, n) < 1, \mu_a(M, \pi, n) > 0\}$.

Corollary 2 *Any auction mechanism M with a nontrivial reserve price, yielding $\pi(M) \in (0, 1)$, is an expected-revenue-inferior mechanism for seller to adopt.*

Though the proof (in Appendix C) conveys little insight, Corollary 2 is quite intuitive. There is an unavoidable detriment in this model, whenever $\pi < 1$. That is, with probability $(1 - \pi)^N$, the independent mixed-strategy decisions lead to no potential bidder participating, which is a cost that both seller and a social planner would take into account. (Appendix B considers an alternative model avoiding this detriment.)

[23] Indeed, with some additional notation, one can readily build an extension of this model featuring a number N_L of potential bidders who face information-acquisition costs c_L and N_H who face costs $c_H > c_L$. In such a model, compare any auction mechanism M_1 for which the high-cost potential bidders participate with positive probability (in which case, all low-cost potential bidders strictly prefer to participate) with any mechanism M_2 for which the high-cost potential bidders do not participate. It is straightforward to show that a seller will prefer M_1 to M_2 if and only if an efficient social planner does. Cox, Dinkins and Swarthout [2001] outline a model in which each potential bidder draws his information-acquisition costs from a smooth distribution.

Should a nontrivial reserve price be used, a further detriment that is an inefficiency is introduced: not only is there no sale with probability $(1-\pi)^N$, there is also a probability

$$\sum_n \sum_{a=0}^{n} \left[1 - s\left(M, a, \pi\left(M\right), n\right)\right] \binom{n}{a} \mu_a\left(M, \pi\left(M\right), n\right) \beta_n \tag{4}$$

that one or more potential bidders participate, but none of them are willing to pay the reserve price.

When the number of bidders responds endogenously to the profitability of competing, there is no counterbalance to make up for the loss of a sale due to a nontrivial reserve price. Occasionally, a reserve price would have, for example, fallen between the highest and second-highest bids in a second-price auction, or prevented a single participant from obtaining the asset for merely the entry fee, but the increased revenue such events create will have been taken into account in bidders' calculations of the probability with which to participate. Levin and Smith [1994] find that a nontrivial reserve price enhances revenue in common-value auctions with entry; their result is entirely due to disallowing entry fees, disclosure of seller's information, and other surplus-extracting devices that shed the revenue losses in (4).[24]

Via Corollary 2, endogenizing bidder participation turns much of standard auction theory on its head. A reserve price (or bidder-specific reserve prices if bidders draw types asymmetrically) is the focus of Myerson's [1981] original "Optimal Auction Design" paper, and of much of the optimal auctions and mechanism design literature since (see, for examples, surveys in Klemperer [2000] and Krishna [2002]). Indeed, auction policy papers also focus on the reserve price as if it were a key variable (Klemperer [2002]). Yet when the number of bidders becomes an endogenous variable, the

[24] Levin and Smith [1994] have a more primitive device in their model that they call an entry fee, but it is an information fee, in that it must be paid before bidders learn their signals; it in essence allows seller to employ a lump-sum tax on participants before they become privately informed. They obtain the result that a nontrivial reserve price is called for when they assume this information fee has to be set to zero. This paper follows a tradition in the literature, led by Cassady [1967], Milgrom and Weber [1982] and Samuelson [1985] in the normal usage of the term entry fee (as a fee incurred after participants become privately informed).

It also follows the tradition in the optimal auctions literature, and indeed in auction theory more generally (the only other exceptions I know of are McAfee and Reny [1992] and Crémer, Spiegel and Zheng [2009]), of assuming that any surplus-extracting device is potentially distortive, and thus ruling out devices that are in essence lump-sum taxes. I thank Jeroen Swinkels for emphasizing this issue, and for pointing out that a limit to the generality of this paper's results is that they apply only after a seller has exhausted usage of devices that are essentially lump-sum taxes.

Levin and Smith criticize Samuelson [1985] for considering the impact of entry fees in a model where the aggregate expenditures on becoming privately informed are exogenous. The current model withstands that criticism.

Chakraborty and Kosmopoulou [2001] report a similar characterization to Corollary 2, that with entry, an auction with a nontrivial reserve price is revenue-inferior to some auction with a lower reserve and an entry fee. It is not clear what model of entry yields the nonstochastic number of participants in their paper, which depends on a screening level.

reserve price becomes a uniquely inferior tool for extracting surplus from bidders; a rational seller does not use it, and an efficient social planner is glad he doesn't.

7 Revenue and Participation

In view of Corollary 2, the remainder of the text limits mechanisms to $M \in \mathbb{M}^Z = \{M \in \mathbb{M} \,|\, r = 0\}$, zero-reserve-price auctions. This section shows that seller's announcement of M affects expected revenue solely through its effects on the participation probability π. To develop and understand this result, I begin with some natural comparative statics: two auction mechanisms with the same π have the same expected revenue, and a change in mechanism which would lead to a higher expected revenue for any exogenously given number of bidders will lead to a lower π. Formally,

Proposition 1 *For any* $\{M, M'\} \subset \mathbb{M}^Z$*,*
[i]: $\{\pi(M) = \pi(M')\} \Rightarrow \{R(M) = R(M')\}$*;*
[ii]: $\{\pi(M) = \pi(M')\} \Rightarrow \{\overline{V}(M) - b\overline{a}(M) = \overline{V}(M') - b\overline{a}(M')\}$*;*
[iii]: $\{\mathcal{R}(M,n) \gtreqless \mathcal{R}(M',n) \, \forall n \in \mathbb{N}\} \Rightarrow \{\pi(M) \lesseqgtr \pi(M')\}$*.*

The proof is in Appendix C.[25]

Proposition 1 implies that revenue consequences of an increased number of bidders are necessarily less rosy when the extra bidders arrive via a rational participation calculation:[26]

Remark 1 *Suppose a seller can switch to an auction mechanism that increases equilibrium participation. Then each participant has a lower chance of winning, and so in equilibrium requires a higher expected profit in the event of winning. The winner's higher expected profit means an expected revenue* further below *expected value transferred.*

A host of econometric studies of auction markets are not sensible in this context, cases where revenue, the high bid, or some similar variable is estimated using the number of bidders as an exogenous explanatory variable. If ten potential bidders decided to participate expecting about three participants, but mixed strategy participation decisions happened to lead to six showing up, no wonder the extra bidders led to higher revenue: the auction rules were sufficiently extractive

[25] Proposition 1 applies only when $\pi(M)$ is a function. The only auction mechanism this appears to rule out is the McAfee, McMillan and Reny [1989] mechanism that extracts full surplus if $n \geq 2$ bidders are exogenously given. I cover the details of comparison with that mechanism in footnote 29.

[26] This is the intuition behind the negative answer in Harstad [2008].

of bidders' surplus that no one wanted to be a fourth bidder. It would be interesting to discover the circumstances under which a higher *expected* number of bidders was associated with a higher expected price, but the historical record of auctions (outside carefully designed laboratory experiments, cf., e.g., Levin and Smith [2002]) does not include data on the expected number of bidders. The actual number of bidders is no substitute.[27] *That revenue is higher when the* realized *number of bidders is higher does not imply that a seller prefers to take steps to increase the* expected *number of bidders.*

For many empirical studies, especially merger-and-acquisition studies, it remains a problem that the record does not indicate the number of participants, but at most the number of actual bidders. Hence, the size of a winner's curse adjustment a bidder ought rationally to make depends on a variable (or inferences about that variable) unavailable to the empirical analyst.

Proposition 1.[*ii*] finds the difference $\overline{V}(M) - b\overline{a}(M)$ takes on the same value for any mechanisms M, M' that attain the same π; let $W(\pi)$ denote this difference. Then let a simple function R on the entire unit interval be defined by

$$R(\pi) = W(\pi) - c\pi N. \tag{5}$$

Corollary 3 *$R(M) = R[\pi(M)]$ for all $M \in M^Z$; that is, $R(M)$ can be projected onto $[0,1]$ to yield $R(\pi)$.*

Proof. *Proposition 1 implies that M influences $R(M)$ only through its influence on $\pi(M)$. Theorem 3 shows that the entire interval $[0,1]$ can be reached.* ■

Remark 2 *Corollary 3 finally changes auction theory's view of the comparative roles of the various surplus-extracting devices available to a seller, from complements (their role in exogenous-number-of-bidders models in the tradition of Milgrom and Weber [1982]) into their common-sense role of substitutes. With exogenous n, a seller who had introduced some subset of: switching to an English auction, releasing public information, setting an entry fee, and adding a nontrival reserve price, would still gain by incorporating the remaining surplus-extracting devices. With endogenous*

[27] If a historical series of auctions arguably results from the same equilibrium for each auction, then the binomial distribution that is the number of participants can be estimated from the series. Many empirical auction databases, however, arise from situations far enough from ex-ante symmetry to warn against direct application of this model. Nonetheless, observed data on the number of bidders may incorporate entry decisions based on which rivals were expected to take part with what probabilities. If the identities of participants are recorded in the database, a separate binomial distribution representing participation of each potential bidder could be estimated.

participation, it is (solely) the equilibrium probability π that interests seller, and alternative methods of accomplishing an improvement in this variable are substitutes for each other.

8 The Content of Optimal Auctions

Theorem 3 and Corollary 3 imply that $R(\pi)$ in (5) is continuous. As its range is obviously bounded, it attains a maximum. Let $R^* > 0$ be the maximum attainable level of revenue. Since a screening level is impossible, it is unsurprising that I have not been able to demonstrate strict concavity of $R(\pi)$. Accordingly, there may be multiple values of π attaining R^*; let $A = \{\pi \in [0,1] \mid R(\pi) = R^*\}$. By continuity, A contains a minimal and a maximal element, π_0^* and π_1^* (not necessarily different). As $R(0) = 0$, $\pi_0^* > 0$.

Proposition 2 *If $N > 2$, $\pi_1^* < 1$.*

Proof. Evaluation of (5) as π decreases from 1 to $1 - \triangle\pi$ shows a revenue gain from decreased information-acquisition costs that is linear in $\triangle\pi$, and a revenue loss from a probability of 0 participants that is of the order $(\triangle\pi)^N$. ■

Proposition 2 applies to $N = 2$ potential bidders in a common-value auction.[28] In that case, it is revenue-inferior to adopt an auction that leads to both bidders participating with probability one. The seller will have some revenue-superior alternative which will lead potential bidder 2 to be indifferent over participating even when he infers that there will be at least a $(1 - \pi_1^*) > 0$ probability of facing no competition.

A little structure enables characterizing the size of the set of optimal auctions. As mentioned in section 3, without loss of generality, the set $\mathbb{M}$ of seller's feasible auction mechanisms can be embedded in a real space, $\mathbb{M} \subset \Re^{d+1}$. Then the set of zero-reserve-price auctions can be viewed as one dimension smaller (by omitting the $r = 0$ component): $\mathbb{M}^Z \subset \Re^d$, let $D = \{1, \ldots, d\}$. D can be called the set of components the seller chooses.[29]

Define $\mathcal{M}^C$ to be the set of auction forms m for which some component $i_m \in D$ continuously alters π and allows spanning $[0,1]$. Theorem 3 above shows that $\mathcal{M}^C \neq \varnothing$; the continuity used

[28] Harstad [2008] proves and interprets this result in a somewhat simpler model.

[29] The exact dimensionality depends on modeling choices (as to what constitute the components) that otherwise distract from the paper. The entry fee is one component. Whether n and a are revealed generates two more. At least one component could be generated by whether the auction is dynamic (if the degree of information dispersal during the course of the auction is an issue, more than one component), and still first-price and second-price auctions have not been distinguished. Mares and Harstad [2003] show that seller's information disclosure options cannot be fully specified via a single component.

in that proof is known to hold for the English and first-price auctions (and for public information disclosure). Next, define a set of auction mechanisms $\underline{\mathbb{M}} = \left\{M \in \mathbb{M}^Z \,|m \in \mathcal{M}^C\right\} \subset \Re^d$; this is the set of mechanisms selling without reserve for which the auction form exhibits continuity and spanning via some component of D.

Consider $\underline{\mathbb{M}}$ the domain of choice for seller, in light of Corollary 2. Without loss of generality, the d components of D can be ordered so that component $i \in \Delta = \{1, \ldots, d^*\}$ $(0 < d^* < d)$ denotes a spanning, continuous component for all $m \in \mathcal{M}^C$ (φ is one such component, guaranteeing $0 < d^*$; other examples are provided below). For arbitrary auction mechanism $M \in \underline{\mathbb{M}}$ and arbitrary component $\delta \in \Delta$, let $\underline{\mathbb{M}}^\delta|_M \subset \underline{\mathbb{M}}$ denote the set of mechanisms which differ from M only in the value of the component δ (thus, for example, if δ connotes the entry fee, all mechanisms in $\underline{\mathbb{M}}^\delta|_M$ differ only in the entry fee, so if M incorporates the auction form m for which only the number of actual bidders is learned before bidding under first-price rules, then this holds for all auctions in $\underline{\mathbb{M}}^\delta|_M$). Let $\mathbb{M}^* = \{M \subset \underline{\mathbb{M}} \,|\, \pi(M) \in A\}$, a collection of optimal auctions.[30] Then,

Theorem 4 *(The "Content" Theorem): For arbitrary auction mechanism $M \subset \underline{\mathbb{M}}$ and arbitrary component $\delta \in \Delta$, $\left(\underline{\mathbb{M}}^\delta|_M \cap \mathbb{M}^*\right) \neq \varnothing$. That is, any auction in $\underline{\mathbb{M}}$ can be converted into an optimal auction in $\mathbb{M}^*$ merely by adjusting* any one *component in Δ.*

Proof. Select an arbitrary $M_0 \in (\underline{\mathbb{M}} \backslash \mathbb{M}^*)$. If $\pi(M_0) \in A$, nothing remains to be proven, so assume $\pi(M_0) \notin A$. Select an arbitrary component $\delta \in \Delta$. Construct $\widehat{M} \in \underline{\mathbb{M}}^\delta|_M$ by changing M_0 solely in component δ, as follows. Take an arbitrary $\pi^* \in A$; if $\pi(M_0) < \pi^*$, set $\pi(\widehat{M}) = 1$ (by $\varphi = -c - b$, for example), else if $\pi(M_0) > \pi^*$, set $\pi(\widehat{M}) = 0$ (by $\varphi = E[T]$); $\delta \in \Delta$ insures this is possible. As in Theorem 3, continuity implies the existence of a value for this δ^{th} component yielding $M^* \in \mathbb{M}^*$, with M^* differing from M_0 only in this δ^{th} component.[31] ■

[30] Some readers may question how these optimal auction mechanisms compare to the mechanism which extracts full surplus in McAfee, McMillan and Reny [1989]. For their mechanism, call it M^{MMR}, the unique equilibrium continuation is $\pi\left(M^{MMR}\right) = 0$, hence $R\left(M^{MMR}\right) = 0$. To arrive at a sensible comparison, consider mechanisms M_φ^{MMR} with negative entry fees appended. Setting $\varphi < -c - b$ necessarily generates a revenue-inferior auction; the reverse inequality suffers the same $R\left(M_\varphi^{MMR}\right) = 0$ problem as M^{MMR}. So consider $\varphi = -c - b$: for M_{-c-b}^{MMR}, every $\pi \in [0, 1]$ is an equilibrium continuation. Selecting $\pi \in (\pi_1^*, 1]$ yields excessive incurrence of information-acquisition costs with no compensation; selecting $\pi \in [0, \pi_1^*]$ runs into the same problem as a reserve price: there is an excess probability of no sale (happening anytime $a < 2$), with no compensation. So any equilibrium selection π is revenue-inferior to the second-price auction $\overline{M}_\varphi$ of Theorem 3 that attains the same π, hence suboptimal. If a mechanism similar to Crémer and McLean [1985], [1988] were to apply to a common-value auction, it would suffer the same problems. So would the mechanism of McAfee and Reny [1992], which also depends on using information fees, ruled out here (cf. footnote 23).

[31] A geometric interpretation: construct the projection mapping $proj_\delta : \Re^d \to \Re^{d-1}$ by deleting the component corresponding to an arbitrary $\delta \in \Delta$ from any d-vector $M \in \underline{\mathbb{M}}$. Then the proof of Theorem 4 has shown for every $M \in \underline{\mathbb{M}}$ an element $M_M^* \in M^*$ such that $proj_\delta(M) = proj_\delta(M_M^*)$. Note that M^* is typically a strict superset of the set thus obtained.

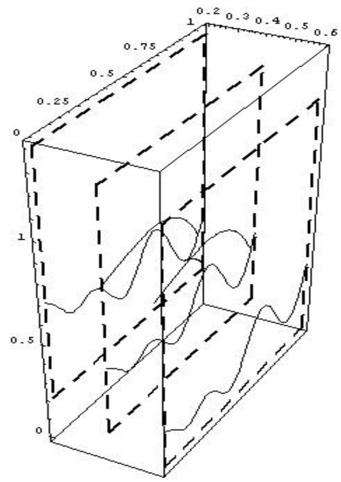

Figure 2: The Set of Optimal Auctions

Thus, prior optimal-auction characterizations depend critically on the implicit assumption that a seller has a captive audience: there will be exactly n bidders no matter how the seller changes auction rules to extract more surplus. When the number of bidders responds endogenously to the profitability of competing, the content of optimal common-value auctions is merely this: choose *any* auction form, commit to sell without reserve, and adjust *any* continuous parameter to avoid overly encouraging or overly discouraging bidder participation.

Figure 2 may help to visualize Theorem 4. It simplifies by imagining that $\underline{\mathbb{M}}$ has three dimensions: φ on the vertical axis, plus one dimension in which $\underline{\mathbb{M}}$ can take on one of three discrete values (e.g., English, second-price, or first-price auction form, for a seller we imagine to be constrained to those three choices), and one dimension in which a variable can be chosen over an interval, but may not necessarily span the range of π. The set $\underline{\mathbb{M}}$ is then the union of three rectangles in parallel vertical planes, outlined in Figure 2 by dashed lines. The set of optimal auctions $\mathbb{M}^*$ is represented by the union of a collection of curves shown lying in the three rectangles. The optimal auctions $\mathbb{M}^*$ span $\underline{\mathbb{M}}$ in that, from any point in one of the three rectangles, it is possible to reach one of the curves by moving only vertically, that is, by adjusting only the entry fee. (To be consistent with the Theorem, the curves must, collectively, contain continuous paths from the left to the right edges of all three rectangles.)

A variety of surplus-extracting devices might exhibit sufficient continuity to apply this logic. For example, suppose seller observes X_{N+1} (which is affiliated with asset value T), and consider mechanisms M_y^1, all first-price auctions with $\varphi = \varphi_0$ (arbitrary), $r = 0$, and with seller making a public announcement of $Z_y = X_{N+1} + y\zeta$, where ζ is an independent standard normal (white noise), and y a scalar parameter of the noisiness of this public announcement. Then (by Theorem 17 in Milgrom and Weber [1982] and Proposition 1.[*iii*] above), $\pi\left(M_y^1\right)$ is nondecreasing in y; assume (naturally) that it is continuous. Suppose $\pi\left(M_0^1\right) < \pi_0^* < \pi\left(M_Y^1\right)$ for sufficiently large Y. That is, full and honest public announcement of seller's information is overly extractive of surplus, but not a public announcement where the signal-to-noise ratio is very small (this is, in essence, a spanning supposition). Then the argument of Theorem 4 can be applied to derive the existence of a y^* such that $M_{y^*}^1$ is an optimal auction.[32]

For another example, let $\widehat{M_\delta}$ denote an auction (with seller's information-disclosure policy and $\varphi = \varphi_0$ arbitrary) in which the highest sealed bid win at a price $p = \mu b_{(1)} + (1-\mu)\, b_{(2)}$, a convex combination of the two highest bids. Naturally assuming continuity in μ, if $\pi(M_2) < \pi_1^* < \pi(M_1)$ (a spanning assumption that the first-price but not the second-price auction is insufficiently extractive of surplus given the arbitrary entry fee and information policy), then only the weight μ on the highest bid needs to be adjusted to obtain an optimal auction.[33]

In a world where a wide variety of auction mechanisms are employed by experienced and apparently successful auctioneers and frequent auction sellers or bid-taking procurers, Theorem 4 has the comforting conclusion that this variety is not *per se* unambiguous evidence that some of these sellers and auctioneers must be choosing suboptimally. If some aspect of a particular situation falling outside the model creates a preference for one auction form over another, nothing in the model surmounts that preference, so long as some variable remains sufficiently adjustable.

The sharpness of Theorem 4 stems partly from the exactness attained via a mixed-strategy participation decision arrived at simultaneously by ex-ante symmetric potential bidders (thus not yet privately informed). Appendix B finds the bulk of these results attainable, if granted a sufficiently useful equilibrium selection, when potential bidders sequentially decide whether to participate. The intuition is this: if behavior once participating is symmetric, and one potential bidder who is indifferent over whether to participate does take part, then that one participant's indifference drives

[32] A similar example: some sellers categorize assets being sold, so each asset in a category has an appraisal in a given range (e.g., \$20K-\$40K for one range, \$40K-\$80K for the next). For any auction form with the property that a broad enough range is insufficiently extractive and an exact announcement of the appraisal overly extractive, there exists a range width yielding an optimal auction. Thus, from an arbitrary auction form, only the range width of this categorization need be altered to obtain optimality.

[33] The same analysis would apply if μ were the probability that a first-price auction would be run, if seller had precommitted to a rule wherein each bidder submitted both a bid for a first-price and another bid (presumably distinct) for a second-price auction, before a credible randomization determined which set of bids would be used.

revenue whether his participation decision was made simultaneously or sequentially.

9 Concluding Remarks on Generality

The contrast is striking: Many papers calculate an "optimal auction," having innocuously (or at least without comment) assumed there are n bidders. When this number remains fixed as the role of being a bidder is made far less profitable, these authors are in essence assuming irrational behavior, for most situations where they would have us apply their results. Those results typically find a particular auction form to be optimal, and it typically revolves around strategically setting a reserve price which has a significant chance of preventing a sale.

When bidders are also rational in deciding whether to bid, and the number of bidders is explicitly recognized as an endogenous variable, these results are completely overturned: seller optimally conducts an allocatively efficient auction (even in some situations where this means sale to a bidder other than the highest-valuing participant–because in equilibrium that participant does not pay the entry fee); the only aspect of an auction design that, *per se*, characterizes it as suboptimal is a nontrivial reserve price. Selling without reserve is the full content of optimal auctions when participation is endogenous.

As contemplated in auction theory, a nontrivial reserve price is almost never seen in practice (Cassady [1967]). The contemplated reserve price is a credible binding commitment that, if no bid exceeds it, the asset will not now and never in the future be available to the potential bidders. In some situations, such a commitment may stretch credibility, but in many, I suspect a tool so impacting yet so blunt is not used because it would pointlessly introduce inefficiency. What is common, and in the industry usually called a reserve price, is a price below which the current auction will end without a sale, but the same asset will be put up for sale again later[34] (often,

[34] Though he does not comment on the different usage of the term reserve price, this is the sense in which McAfee [1993] finds a nontrivial reserve price in the steady state of a model where sellers repeatedly hold auctions, with private-values buyers each time deciding which auction to attend. When a reserve price is not met, the seller simply holds another auction, possibly with a different reserve price, in the next period. Successful transactors, buyer and seller, exit the market. Each period, new buyers and sellers are exogenously placed in the market (though all buyers, new or having bid before, choose among sellers' auctions each period).

Though quite different models address quite different issues, this model and McAfee's are strikingly congruent. In particular, his sellers' steady state choices, essentially just in-period reserve prices, are those that a social planner would choose. In his model, a buyer's cost of attending an auction is foregoing opportunities to compete in other auctions, which is an acceptable interpretation of the information-acquisition cost c here. His model explicitly solves for sellers' steady state behavior; here other sellers are implicitly represented via potential bidders only participating if expected profitability is high enough. His bidders decide which auction to attend, but the number of bidders in the marketplace is exogenous; thus, most of the questions addressed here cannot be addressed in his model.

there may be negotiations between the seller and the high bidder to buy between the final bid and the reserve price). Such a policy, of course, limits potential inefficiencies to a wholly lower order of magnitude; introducing it would bring complications of dynamic negotiations into the model, which I have avoided.

This paper has affirmed how natural and robust are the two principal result reversals when the expected number of bidders is endogenous: optimal auctions do not generate inefficiencies, and do not set a nontrivial reserve price. The principal result is that while selling without reserve, the set of optimal auctions is large, consisting of single-parameter adjustments of all auctions. While the demonstration has hopefully been as straightforward as possible, it is clear that the characterizations allow several natural extensions:

- The oft-seen formulation of an auction problem as an abstract mechanism design problem contemplates payments to or from losing bidders. Such payments are easily incorporated here, although Theorem 3 and Proposition 1 render them pointless.
- The bid-preparation cost above was exogenous, and independent of the form of the auction. If strategic issues were to make it more costly to prepare a bid in a first-price auction than in a second-price or an English auction (due to some variant on incentive compatibility), a more complicated twist on the tools provided here would be needed.[35]
- The assumption of a single asset for sale does not seem critical to the qualitative results. However, the ease with which extension to the modal multiple-unit auction model (where each bidder can acquire but one asset) arises in Milgrom [1981] and Pesendorfer and Swinkels [1997] would be somewhat misleading. If k identical assets are sold, the probabilities of $0, \ldots, k-1$ actual bidders would significantly clutter up the expected revenue formula.

[35] Engelbrecht-Wiggans [2001] argues that the strategic simplicity of English auctions, in some simple settings, yields lower bid-preparation costs than first-price sealed-bid auctions, and so English auctions might attract more bidders and attain higher expected revenue. He demonstrates the possibility in a simple example with independent, uniformly distributed private values and infinitely many potential bidders. Unfortunately, the notion of strategic simplicity does not admit nearly as facile a sensible definition when going beyond such a simple setting. In particular, it may be that an English auction becomes far less simple strategically when a significant entry fee is prepended. (Engelbrecht-Wiggans does not write as if he is comfortable with the notion that an English auction necessarily remains strategically simpler once common-value elements enter the model.) Since in the equilibrium above, aggregate expected bid-preparation costs fall on the seller (Theorem 2), this by itself gives an incentive to favor devices which extract surplus while facing bidders with lower bid-preparation costs. For a given π, the mechanisms with lower bid-preparation costs yield higher revenue. For there to be a sufficiently large set of such lower-bid-preparation-cost mechanisms to span continuously the range of participation probabilities, and thus render all higher-bid-preparation-cost mechanisms inferior, is likely to depend on some controversial assumptions about strategic simplicity.

- The model has been designed so that adding other sellers who are auctioning related assets is virtually automatic. (No problems are created if one seller's auction exhibits an information-acquisition cost of c, while a possibly more distant seller has a cost $c' > c$.)

A nontrivial dynamic structure would, however, introduce concerns not yet addressed. Among them, both sellers and potential bidders may have incentives to invest in building reputations. Nonetheless, a stride in this direction is made here: if an analysis of such reputational issues is to be applicable to markets where a subset of firms in an industry appear as bidders, reputational investments need to be viewed in terms of their discounted expected profitability when responses of other players include an endogenous decision as to whether and when to play.

This model assumes rational behavior consistent with a symmetric equilibrium. Asymmetric equilibria at the bidding stage are certainly not going to be unique, so it is unclear how to prepend an entry stage, without a unique expected profitability calculation. Asymmetric participation decisions are presumably rife for signaling a preferred asymmetric equilibrium. A "symmetric sequential" entry model which then assumed symmetric behavior following sequential entry decisions can be built; it yields similar but less sharp results. An outline is provided in Appendix B.

Laboratory evidence suggests the winner's curse is not easily overcome in common-value auctions (Kagel, Levin and Harstad [1995]); however, it is far from clear how to model participation decisions of potential bidders who will not follow up by bidding rationally. Nor can I envision how to model usefully the participation decision of a potential bidder who will himself behave rationally, but who cannot predict even the number of irrationally-behaving rivals who will participate.

10 Appendix A: Mathematics of the Participation Decision

Recall that N is the exogenous number of potential bidders, $\pi(M)$ the symmetric equilibrium probability of participating given mechanism M, and n and a numbers of participants and of actual bidders. Throughout, the usual binomial formula for the probability of z successes in Z trials, each with independent success probability ς is denoted

$$\beta\left(z, Z, \varsigma\right) = \binom{Z}{z} \varsigma^{z} \left(1 - \varsigma\right)^{Z-z}.$$

Thus, if N potential bidders each participate with probability π, the probability of n participants is $\beta\left(n, N, \pi\right)$. A potential bidder analyzing the consequences of proceeding to the next step of the game (participating or actually bidding) rationally evaluates the likelihood of different numbers of rival competitors according to $\beta(n-1, N-1, \pi)$, which accounts for the presumption that he (the analyzing potential bidder) proceeds–even if this is not a certainty, all behavior is otherwise payoff-irrelevant.[36] When context makes clear, I will shorten $\beta(n, N, \pi)$ to β_n and $\beta(n-1, N-1, \pi)$ to β_{n-1}. When an arbitrary potential bidder i becomes a participant, I will harmlessly treat the renumbering function $ren(i, n, N)$ that would provide his position in the numerically ordered set of participants as if it were the identity function, and refer to the continuing roles of the player who begins as potential bidder i as if he becomes participant i if he participates, and actual bidder i if he pays the entry fee. Symmetry attained through $\mathcal{A}.1$ and $\mathcal{A}.6$ allows a focus throughout on potential bidder 1, participant 1, and actual bidder 1.

At the point that the decision to become an actual bidder (to incur the bid-preparation cost b and pay the entry fee φ) is made, participant 1 has observed estimate $X_1 = x$. Two cases must be developed.

Case 1: $M \in \mathbb{M}^K$, where $\mathbb{M}^K$ is the subset of mechanisms for which auction form m specifies that participants know (perhaps because the seller informed them, perhaps because the seller could not prevent their knowing) the number of participants, n, before deciding whether to pay the entry fee.

Let Ψ denote the set of η_1-measurable subsets of $\mathcal{X}$, with element $\psi \subset \mathcal{X}$, and $\Psi_x = \{\psi \in \Psi | x \in \psi\}$. Let $\rho(\cdot) : \Psi_x \times \mathcal{X} \to [0, 1]$ be defined by $\rho(\psi, x) = \Pr_\eta[X_2 \in \psi | X_1 = x]$, the probability that a given rival participant observed an estimate in the set ψ, conditional on participant 1 observing estimate x. Next define the binomial distribution $B(\psi, x, n)$ attaching density $\beta[j-1, n-1, \rho(\psi, x)]$ to values $j = 1, \ldots, n$. Let $\xi^K(M, a, n, \psi, x)$ be the expected profitability (gross of bid-preparation cost and entry fee, but net of information-acquisition cost) of actually bidding in auction M, when there are a actual bidders, n participants, the a actual bidders all observed estimates in ψ, and actual bidder 1 observes $X_1 = x$. Define for $\psi \in \Psi_x$

$$\overline{\xi}^K(M, n, \psi, x) = \sum_{j=1}^{n} \xi^K(M, j, n, \psi, x)\, \beta[j-1, n-1, \rho(\psi, x)],$$

which is expected profitability when the number of actual bidders is determined by the number of rival participants who observe estimates in ψ.

[36] This insight is originally due to Matthews [1987] (who is credited in McAfee and McMillan [1987a]), and is employed in Harstad, Kagel and Levin [1990]. In all three papers, the uncertain number of bidders follows an exogenous distribution.

Next let $\tau^K(M,n,\varphi,\psi,x) = \left\{\varsigma \in \Psi_x | x \in \varsigma \Rightarrow \overline{\xi}^K(M,n,\psi,x) \geq \varphi + b\right\}$, so that the correspondence $\Upsilon_{M,n,\varphi,x}(\cdot) : \Psi_x \to \Psi_x$ can be defined by

$$\Upsilon^K_{M,n,\varphi,x}(\psi) = \left\{\varsigma \in \Psi_x | \varsigma \in \tau^K(M,n,\varphi,\psi,x)\right\}.$$

What Υ^K does is take an arbitrary subset ψ of the space of estimates, a subset consistent with participant 1 observing $X_1 = x$, use it to find out expected profitability if rival participants pay the entry fee φ iff they observe an estimate in ψ, and then map that expected profitability into the subsets of Ψ_x for which paying the entry fee is rational (i.e., yields an expected profitability preferable to ceasing further competition). Finally,

$$\begin{aligned} \Xi^K(M,n,\varphi) &= \left\{x \in \mathcal{X} | \exists \psi \in \Upsilon^K_{M,n,\varphi,x}(\psi) \,|\, \{\exists (\psi\prime, z) \in (\Psi_x \times \mathcal{X}_1) \,|\psi \subset \psi\prime, z \in int(\psi\prime \backslash \psi)\} \Rightarrow \psi\prime \notin \Upsilon_{M,n,\varphi,x}(\psi\prime) \right. \\ \alpha^K(M,n) &= \int_{\Xi^K(M,n,\varphi)} d\eta_1(x). \end{aligned}$$

Here, $\Xi^K(M,n,\varphi)$ is the maximal subset of $\mathcal{X}$ consisting of those estimates with a sufficiently high interim expected profitability to justify continuing to compete in auction M with $n-1$ rival participants, and $\alpha^K(M,n)$ is the ex-ante probability that a potential bidder whose action is to participate will end up becoming an actual bidder.

Continuing with case 1, $M \in \mathbb{M}^K$, define, for $n = 1,\ldots,N, i = 1,\ldots,n$, the event that participants $1,\ldots,i$ observe estimates leading them to continue, while participants $i+1,\ldots,n$ observe estimates leading them to cease competing:

$$\Gamma^K(M,i,n) = \left[\left\{X_j \in \Xi^K(M,n,\varphi)\right\} \Leftrightarrow \{j \leq i\}, j = 1,\ldots,n\right].$$

Let

$$\begin{aligned} \mu^K(M,i,n) &= \Pr_\eta\left[\mathbf{X}_n \in \Gamma^K(M,i,n)\right], \text{ and} \\ \lambda^K(M,i,n) &= \Pr_\eta\left[\mathbf{X}_n \in \Gamma^K(M,i,n) \,|X_1 \in \Xi^K(M,n,\varphi)\right] \end{aligned}$$

denote the probability of this event, and its probability conditional on participant 1 observing an estimate leading to continued competition. By Bayes' Formula, for $n = 1,\ldots,N, i = 1,\ldots,n$,

$$\mu^K(M,i,n) = \lambda^K(M,i,n)\,\alpha^K(M,n). \tag{7}$$

Case 2: $M \in \mathbb{M}^U = \mathbb{M} \backslash \mathbb{M}^K$, when the auction form m specifies that the number of participants is unknown when deciding whether to pay the entry fee.

Let $\xi^U(M, a, \pi, \psi, x)$ be the expected profitability (gross of bid-preparation cost and entry fee, but net of information-acquisition cost) of actually bidding in auction M, when there are a actual bidders, $N-1$ rival potential bidders participated with probability π, the a actual bidders all observed estimates in ψ, and actual bidder 1 observes $X_1 = x$.[37] Define for $\psi \in \Psi_x$

$$\overline{\xi}^U(M, \pi, \psi, x) = \sum_{i=1}^{N} \sum_{j=1}^{i} \xi^U(M, j, i, \psi, x)\, \beta(i-1, N-1, \pi)\, \beta[j-1, i-1, \rho(\psi, x)].$$

The next four steps of case 2 correspond exactly to those of case 1, substituting $\overline{\xi}^U$ above for $\overline{\xi}^K$ to define $\tau^U(M, \pi, \varphi, \psi, x)$, τ^U to define $\Upsilon^U_{M,\pi,\varphi,x}(\psi)$, Υ^U to define $\Xi^U(M, \pi, \varphi)$, Ξ^U to define $\alpha^U(M, \pi)$. In case 2, the event that participants $1, \ldots, i$ actually bid, and $i+1, \ldots, n$ cease competing:

$$\Gamma^U(M, \pi, i, n) = \left[\left\{X_j \in \Xi^U(M, \pi, \varphi)\right\} \Leftrightarrow \{j \leq i\},\ j = 1, \ldots, n\right],$$

which is well-defined although the participants do not know that they number n. The probability of the first i participants becoming the only actual bidders, given that N potential bidders each become a participant with probability π, must take the probabilities of events $\Gamma^U(M, \pi, i, n)$ and weight them according to their likelihood:

$$\begin{aligned} \mu^U(M, \pi, i) &= \sum_{n=1}^{N} \beta(n, N, \pi) \Pr\left[\mathbf{X}_n \in \Gamma^U(M, \pi, i, n)\right], \text{ and} \\ \lambda^U(M, \pi, i) &= \sum_{n=1}^{N} \beta(n, N, \pi) \Pr\left[\mathbf{X}_n \in \Gamma^U(M, \pi, i, n) \,|\, X_1 \in \Xi^U(M, \pi, \varphi)\right] \end{aligned}$$

is the conditional probability given that participant 1 observes a estimate leading to continued competition. As before,

$$\mu^U(M, \pi, i) = \lambda^U(M, \pi, i)\, \alpha^U(M, \pi). \tag{8}$$

[37] If the auction form m specifies that information disclosed by seller is disclosed before the entry fee is paid, ξ^K and ξ^U are specified with respect to the equilibrium in the bidding subgame corresponding to the information actually disclosed. If m specifies that information disclosed by seller is disclosed after the entry fee is paid, ξ^K and ξ^U are specified with respect to the distribution of equilibria expected given the prior distribution of seller's information.

Conclusion 1 *The relevant ex-ante probabilities combine the two cases :*

$$\alpha(M,\pi,n) = \begin{cases} \alpha^K(M,n), & M \in \mathbb{M}^K, \\ \alpha^U(M,\pi), & M \in \mathbb{M}^U, \end{cases},$$

$$\mu_i(M,\pi,n) = \begin{cases} \mu^K(M,i,n), & M \in \mathbb{M}^K, \\ \mu^U(M,\pi,i), & M \in \mathbb{M}^U, \end{cases}, \text{ and}$$

$$\lambda_i(M,\pi,n) = \begin{cases} \lambda^K(M,i,n), & M \in \mathbb{M}^K, \\ \lambda^U(M,\pi,i), & M \in \mathbb{M}^U, \end{cases},$$

with, for any M, each of α, μ_i, λ_i degenerate in one of its last two variables.

Thus, $\alpha(M,\pi,n)$ takes an *ex-ante* view, from the viewpoint of a potential bidder: it is the probability, should he participate, that he will go on to become an actual bidder, evaluated *before* the estimate x is observed. Similarly, $\lambda_a(M,\pi,n)$ is the *ex-ante* probability, should he become an actual bidder, that a potential bidder will find himself to be one of the set $\{1,\ldots,a\}$ actual bidders, and $\mu_a(M,\pi,n)$ is the (unconditional) *ex-ante* probability of $\{1,\ldots,a\}$ being the set of actual bidders.[38]

Note that prior models of endogenous participation have, explicitly or implicitly, assumed a *screening level*: some $\widetilde{x}(M,n,\varphi) \in \mathcal{X}$ such that $\left\{x \in \Xi^K(M,n,\varphi)\right\} \Leftrightarrow \{x \geq \widetilde{x}(M,n,\varphi)\}$. Landsberger and Tsirelson [2000] demonstrate that this is impossible in a common-value auction, for large numbers of potential bidders, under mild assumptions, satisfied by this and most prior models. Whether a corresponding impossibility theorem extends to affiliated-values auctions is unclear. This paper is careful to allow for the fact that $\Xi^K(M,n,\varphi)$ and $\Xi^U(M,\pi,\varphi)$ may not be upper contours of $\mathcal{X}$; much of the complication in cases 1 and 2 above is due to that allowance.[39]

Two cases are also distinguished with respect to actual bidders. An auction form $m \in \mathbb{M}^{K'}$ if the number a of actual bidders becomes known before bidding strategies are selected; let the

[38] What has been shown is that the two extreme possibilities, that participants are informed of their number and that they are not, can be combined into a sensible description of the relevant ex-ante probabilities. This same procedure could readily be followed (with yet more notation) should seller also have some third or fourth options, such as informing participants with some interior probability, or informing them whenever $n > 3$ but not informing them when $n \leq 3$. In other words, the procedures just shown are illustrative, rather than limiting. As stated in the introduction, it is the endogeneity of the degree of competition that changes the picture, not the details of how that is modeled.

[39] A second-price auction with $\varphi = r = 0$ is one special case where Ξ^K and Ξ^U are upper contours of $\mathcal{X}$. In general, the sets $\tau^K(\varphi,\cdot)$ of estimates implying at least φ expected profitability are unions of nondegenerate intervals ("ins") separated by nondegenerate intervals ("outs"). An adjustment improving profitability–decreasing π or φ–continuously expands the "in" intervals. Harmlessly defining the "ins" as closed sets then justifies the reference to maximality of Ξ^K and Ξ^U.

probability of a sale be $s^{K'}(M,a)$, which is the probability that at least one of a actual bidders is willing to pay the reserve price r. For $m \in \mathbb{M}^{U'} = \mathbb{M} \backslash \mathbb{M}^{K'}$, the number of actual bidders is unknown when bidding; let $s^{U'}(M,\pi,n)$ be the probability that at least one actual bidder is willing to pay the reserve price r when either [a] each of n participants becomes an actual bidder iff $X_j \in \Xi^K(M,n,\varphi)$, if $m \in \mathbb{M}^K$ (degenerate in π), or [b] if $m \in \mathbb{M}^U$, each of N potential bidders becomes a participant with probability π, and if a participant, becomes an actual bidder iff $X_j \in \Xi^U(M,\pi,\varphi)$ (degenerate in n). Again, combine these cases via

$$s(M,a,\pi,n) = \begin{cases} s^{K'}(M,a), & M \in \mathbb{M}^{K'}, \\ s^{U'}(M,\pi,n), & M \in \mathbb{M}^{U'}. \end{cases} \tag{9}$$

Notation will be slightly abused when context makes clear by representing this probability as s_r (the reserve price r is the principal component of M affecting this probability).

Getting closer to a characterization: Relying on $\mathcal{A}.4$, let $p(M,a,\pi,n,t)$ be a function indicating the expected price paid by the winning bidder, given auction M, a actual bidders, π probability of participating, n participants, and conditional on a realization t of common trend T. Depending on which cases above apply, $p(\cdot)$ will typically be degenerate in at least one variable. It bears emphasis that $p(\cdot)$ is an *ex-ante* calculation, and thus is symmetric across potential bidders. Define

$$V(M,\pi,n,\varphi,t) = \begin{cases} \int_{\Xi^K(M,n,\varphi)} G^K(M,n,\varphi,x,t)\, v(t,x)\, d\eta_1(x|t), & M \in \mathbb{M}^K, \\ \int_{\Xi^U(M,\pi,\varphi)} G^U(M,\pi,\varphi,x,t)\, v(t,x)\, d\eta_1(x|t), & M \in \mathbb{M}^U, \end{cases} \tag{10}$$

where $G^K(M,n,\varphi,x,t)$ [resp., $G^U(M,\pi,\varphi,x,t)$] is the probability of becoming the winning bidder for a potential bidder who will participate in auction M, when there are $n-1$ other participants [when $N-1$ other potential bidders participate with probability π], the entry fee is φ, he will observe estimate x, and underlying asset value is t. Then $V(M,\pi,n,\varphi,t)$ is the expected asset value to a potential bidder who will participate, conditional on his winning in the circumstances specified by its arguments.

Momentarily assume a potential bidder is one of n participants and one of $a \leq n$ actual bidders; his *ex-ante* expected payoff is

$$\frac{s(M,a,\pi,n)}{a} E_\eta \{V(M,\pi,n,\varphi,T) - E_\eta [p(M,a,\pi,n,\cdot)\,|T=t]\} - \varphi - b - c. \tag{11}$$

In essence, conditioning the price on the common-value component (the inner expectation) makes the outer expectation simply the expected difference between what the winner gets and what he pays for it. The probability that the winner obtains this difference is simply the probability of a sale (s). Ex ante, given a winner, the probability that any one of the a actual bidders is the winner is $1/a$, by $\mathcal{A}.1$ and $\mathcal{A}.6$. For an actual bidder, the bid-preparation cost b, entry fee φ and information-acquisition cost c are subtracted with certainty. (Note that this calculation need not require that the actual bidder know the value of n or a.)

Continuing to assume n participants, the *ex-ante* probability of being an actual bidder is $\alpha(M, \pi, n)$, and of any particular formula (11) being the relevant calculation for an assumed actual bidder is $\binom{n-1}{a-1}\lambda_a(M, \pi, n)$, since there are $\binom{n-1}{a-1}$ ways in which actual bidder 1 could face $a-1$ remaining rivals. Now to step back, assume only that a potential bidder is one of n participants. His expected profit, for $n = 1, \ldots, N$, is

$$w(M, n) = \alpha(M, \pi, n)\left(\sum_a \left[\frac{s_r}{a}E_\eta\{V(\cdot) - E_\eta[p(M, \cdot)|T]\} - \varphi - b - c\right]\binom{n-1}{a-1}\lambda_a(M, \pi, n)\right). \tag{12}$$

Throughout, $\sum_a$ and $\sum_n$ are to be taken as abbreviated forms of $\sum_{a=1}^{n}$ and $\sum_{n=1}^{N}$. Each formula (12), for different n, is relevant (assuming participation) with probability $\beta_{n-1} = \beta[n-1, N-1, \pi(M)]$. Thus,

Conclusion 2 *Equilibrium participation is that $\pi \in (0, 1)$ characterized by*

$$\begin{aligned} 0 &= \sum_n \beta_{n-1} w(M, n) \\ &= \sum_n \beta_{n-1}\left\{\alpha(M, \pi, n)\sum_a \left[\frac{s_r}{a}E_\eta\{V(\cdot) - E_\eta[p(M, \cdot)|T]\} - \varphi - b - c\right]\binom{n-1}{a-1}\lambda_a(M, \pi, n)\right\}, \end{aligned} \tag{13}$$

equating the payoff from nonparticipation to the net expected benefits.

The right-hand side of (13) can be lowered by increasing π. If $\pi = 1$ is allowed as an equilibrium possibility (Proposition 2 finds this a revenue-inferior option for seller), $0 = r.h.s.(13)$ must be replaced by $[r.h.s.(13)] \leq 0 = (\pi - 1)[r.h.s.(13)]$. Equation (13), by implicitly defining the symmetric participation probability function $\pi(M)$, together with equilibrium continuation, provides a complete characterization of potential bidders' behavior.[40]

[40] A corresponding equation is asserted by French and McCormick [1984], and found in simpler models by Harstad

11 Appendix B: An Alternative Sequential-Entry Model

Consider the following "symmetric sequential" participation model. First, the seller announces an auction mechanism M, as above. An exogenous randomization assigns to the N potential bidders a relabeling of their indices, with a potential bidder's realization that he is number i in this relabeling his own private information, and all reorderings equally likely. Then potential bidders are in order given the opportunity to participate (at cost c, as above). As soon as a potential bidder declines to participate (an action that may be the result of a mixed strategy), seller is committed to giving no other potential bidder the opportunity.

As a potential bidder knows the step in this order in which he makes his decision, he knows how many potential bidders have already chosen to participate. As a participant's stage in the order does not get revealed, an opportunity to signal a favorite asymmetric equilibrium via becoming participant 1, for example, is unavailable. After participation decisions have been made, one of the participants may know privately that he is the marginal participant, but none knows the order in which rivals became participants.[41] For symmetric behavior to be possible, the private information of the last participant, as to the equilibrium number of participants, must become public; denote this number $n_e(M)$. Hence only mechanisms where participants learn the number of participants can be considered (in Appendix A, this is the set $\mathbb{M}^K$).

The ex-ante probabilities of a participant becoming an actual bidder, and of participants $1, \ldots, a$ becoming the actual bidders (unconditional and conditional), are exactly the same as $\alpha^K[M, n_e(M)]$, $\mu^K[M, a, n_e(M)]$, and $\lambda^K[M, a, n_e(M)]$, as these terms are deined in Appendix A, and $s(\cdot)$ is unchanged from (9) except that it no longer can depend on π. Lack of dependence on π is also the only change in $p(\cdot)$ above, so the expected profitability of being the $n_e(M)^{\text{th}}$ participant is still (12) above. Hence, $n_e(M)$ is determined by the equilibrium participation constraint

$$w[M, n_e(M)] \geq 0 > w[M, n_e(M) + 1]. \tag{14}$$

For equality in (14), participation by $n_e(M) - 1$ potential bidders with probability 1 and by the $n_e(M)^{\text{th}}$ potential bidder with probability $\pi \in [0, 1]$ are equilibria for all values of π. Selection of the $\pi = 1$ equilibrium (revenue-maximal in this set of equilibria) can be based on it being the

[1990] and Levin and Smith [1994]. The current development is original in extending to affiliated-values settings, avoiding a monotonicity assumption and allowing for the full variety of information flows.

[41] It is solely for this reason that the model builds a counterfactual where a potential bidder's sequence order is his own private infomation.

unique element of this set which is the limit of equilibria for mechanisms differing from M by having infinitesimally smaller entry fees. Of course, virtually as strong a selection argument can be made for the $\pi = 0$ equilibrium, as the unique limit of equilibria for mechanisms differing from M by having infinitesimally larger entry fees. However, usual problems with limits of open sets prevent existence of optimal auctions if the $\pi = 0$ equilibrium is selected. I will just consider the self-servingness of the $\pi = 1$ selection be a weakness of the alternative model, and proceed with it.

Revenue is now $R(M) = \mathcal{R}[M, n_e(M)]$, from (1). Define

$$\vartheta(M, n) = \begin{cases} 1, & n = n_e(M), \\ 0, & \text{otherwise.} \end{cases}$$

This substitutes for the binomial coefficients $\beta(\cdot)$ in the formulas for expected value transferred and the expected number of actual bidders:

$$\begin{aligned} \overline{V}(M) &= \sum_n \sum_a s(M, a, n) E_\eta[T] \binom{n}{a} \mu^K(M, a, n) \vartheta(M, n), \\ \overline{a}(M) &= \sum_n \left\{ \sum_a a \binom{n}{a} \mu^K(M, a, n) \right\} \vartheta(M, n). \end{aligned}$$

Then expected revenue satisfies

$$R(M) \leq \overline{V}(M) - b\overline{a}(M) - cn_e(M), \tag{15}$$

with equality for and only for the selected equilibria attaining equality in (14); let $\mathbb{M}^=$ be the subset of $\mathbb{M}^K$ consisting of those mechanisms M for which equality in (14) and (15) can be attained. Note that $\left(\mathbb{M}^K \backslash \mathbb{M}^=\right)$ contains a dense subset of $\mathbb{M}^K$ (open in the dimensions of $\mathbb{M}^K$ with an interior). Define $\mathbb{M}_{\|n\|} = \left\{M \in \mathbb{M}^K \,|\, n_e(M) = n\right\}$, for $n = 1, \ldots, N$, and $\mathbb{M}^Z = \left\{M \in \mathbb{M}^K \,| r = 0\right\}$.

The following results can be obtained for such a model. [i]. $\{M_0, M_1\} \subset \mathbb{M}^=$ and $n_e(M_0) = n_e(M_1)$ implies $R(M_0) = R(M_1)$. This corresponds to a comparative static of the simultaneous entry model.

[ii]. Suppose a mechanism $M_0 \in \mathbb{M}^=$ with $n_e(M_0) = n_0$ participants in equilibrium. Then there exists $M_1 \in \left[\mathbb{M}_{\|n_0\|} \cap \mathbb{M}^= \cap \mathbb{M}^Z\right]$ (i.e., M_1 does not use a positive reserve price). This M_1 is revenue-maximal in the set $\mathbb{M}_{\|n_0\|}$; revenue comparisons across auction forms for an exogenous number of bidders apply within $\mathbb{M}_{\|n_0\|}$, and surplus-extracting devices are substitutes within $\mathbb{M}_{\|n_0\|}$, with the exception that nontrivial reserve prices are revenue-inferior.

[iii]. Suppose there exist $M_{sm} \in \mathbb{M}^{=}$, $M_{Lg} \in \mathbb{M}^{K}$ such that $1 \leq n_e(M_{sm}) < n_e(M_{Lg}) \leq N$, and $R(M_{Lg}) > R(M_{sm})$. Then there exists $n^* > n_e(M_{sm})$ such that [a] $\mathbb{M}^* = \left[\mathbb{M}_{\|n^*\|} \cap \mathbb{M}^{=} \cap \mathbb{M}^{Z}\right] \neq \emptyset$ (these are all zero-reserve-price auctions attaining equality in (14) for n^* participants), and [b] every auction in $\mathbb{M}^*$ is an optimal auction. Moreover, for an arbitrary auction form m' for which expected profitability is continuous in the entry fee φ, if there exists $M' = (m', \varphi', 0)$ such that $n_e(M') < n^*$, then there exists φ^* such that $(m', \varphi^*, 0) \in \mathbb{M}^*$. In this sense, an arbitrary auction can be made optimal by the change of a single parameter, attaining a quite similar characterization to the principal result of the simultaneous entry model.

[iv]. Let $M_{sm}, M_{Lg} \in \mathbb{M}^{K}$ be such that $n_e(M_{sm}) < n_e(M_{Lg})$. Suppose

$$\begin{aligned} & \sum_a \left\{ s\left[M_{sm}, a, n_e(M_{sm})\right] \binom{n_e(M_{sm})}{a} \mu_a\left[M, \pi, n_e(M_{sm})\right] \right\} \\ \geq & \sum_a \left\{ s\left[M_{Lg}, a, n_e(M_{Lg})\right] \binom{n_e(M_{Lg})}{a} \mu_a\left[M, \pi, n_e(M_{Lg})\right] \right\}, \end{aligned}$$

that is, suppose a sale is at least as likely ex ante under M_{sm} as under M_{Lg}. Then $R(M_{sm}) > R(M_{Lg})$, a sense in which the bidder-discouragement flavor of the simultaneous entry model extends to this model. Note that the sale-probability supposition is critical to result [iv]. (Proofs of these results correspond closely to methods used in the main text and Appendix C.)

Figure 3 illustrates these results, for auctions that use a 0 reserve price. The entry fee φ is shown horizontally, expected revenue R vertically. The solid curve illustrates one type of auction, the dotted curve a second type which extracts less surplus for a given number of bidders. For concreteness, we may call the solid curve English auction revenue, and the dotted curve first-price auction revenue. The vertical line segments on each correspond to values of φ for which the specified auction mechanism lies in $\mathbb{M}^{=}$. In particular, each point in a vertical line segment is revenue associated with one of the multiple equilibria: the lower endpoint is associated with the marginal participant selecting to enter with probability 0, the upper endpoint associated with probability 1.

The rightmost vertical segments are where one participant in an English (solid vertical segment) and in a first-price (dotted) auction is enough to make a second potential bidder indifferent over participating. Along the sloped segment of each curve to the left of its rightmost vertical segment, the second potential bidder strictly prefers to participate, while staying out is the third potential

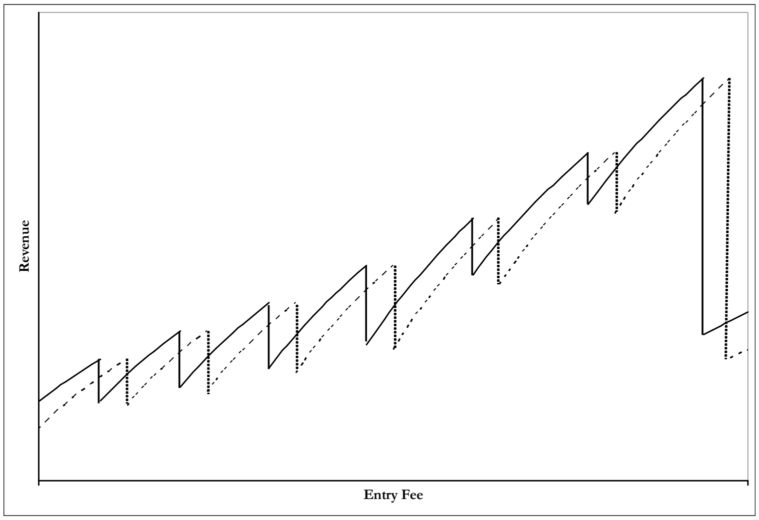

Figure 3: The Sequential Participation Model

bidder's strict preference. Then each curve reaches another vertical segment where the third potential bidder's indifference yields multiple equilibria, followed further left by a sloped segment along which there are three participants.

Each pair of vertical segments corresponding to the multiple equilibria where the i^{th} potential bidder is indifferent over participating peak at exactly the same height. This is a result of equality in (14) and (15). A curve like those shown could be drawn for any auction form; for example, the curve for a second-price auction would have vertical segments that lie between the paired vertical segments shown. The vertical segments shown for the English auction would be shifted to the left if seller's information were publicly disclosed. All such curves for auction forms with 0 reserve prices would reach identical heights at the peaks of vertical segments.

The case illustrated, which would fit a pure common-value environment, will have as an optimal auction (given the self-serving equilibrium selection mentioned above) any auction without a reserve price where the entry fee is set so that the second potential bidder is indifferent over participating. Any auction form which is sufficiently extractive to strictly discourage the second potential bidder via a high enough entry fee will have an entry fee which makes that auction form (with $r = 0$) optimal.

In the general affiliated-values environment, it is possible that the rightmost pair of vertical line segments in Figure 3 do not attain the height of the pair to the left of them. If so, then optimal auctions are those where potential bidder 3 is indifferent over participating but does take part (cf.

Harstad [2008]).

Consider, for an arbitrary auction form m, beginning with n_0 participants in equilibrium, impacts of increasing φ. Increasing from small enough φ, revenue is monotonically increasing, and w, the expected profitability of participating (the l.h.s. of (14)), is monotonically decreasing, while s, the probability of a sale (here, with $r = 0$, the probability that $a > 0$), holds nearly constant. As φ continues to increase, past some level s starts to decrease nonnegligibly. There will be some threshold $\widehat{\varphi}$ at which revenue from n_0 participants hits a local maximum and starts to decrease. Figure 3 is drawn assuming potential bidder n_0 is driven down to indifference over participating before φ reaches $\widehat{\varphi}$.

I know of no assumption on the primitives of the model guaranteeing this will always be the case (this is why the results above in this appendix are stated with such specific conditions). In general, little is known about the behavior of auction mechanisms above $\widehat{\varphi}$. Revenue need not be monotonic in φ above $\widehat{\varphi}$, nor need w be monotonic. It is the case that, for $(m, \varphi', 0) \in \mathbb{M}^{=}$, revenue approaches $R(m, \varphi', 0)$ from below as φ approaches φ' from below. Also, $(m, \varphi, 0) \in \mathbb{M}_{\|n_0\|} \Rightarrow \exists \varphi' | (m, \varphi', 0) \in \mathbb{M}^{=} \cap \mathbb{M}_{\|n_0\|}$. However, $\varphi' > \widehat{\varphi}$ will mean multiple local maxima of revenue in φ for given m, across the set of φ for which an equilibrium with n_0 participants is selectable. In the presence of such multiple local maxima, I know of no argument from primitives that implies the global maximum revenue must lie in $\mathbb{M}^{=}$. Should it not, in essence the theory of auctions with an exogenous number of bidders applies.

Several seminar attendees have insistently pursued the following assertion: a seller who (somehow) had a choice between selling via an auction following the "sequential symmetric" entry of this Appendix and via an auction following the "simultaneous symmetric" entry in the main text above would always prefer the former. I first provide a counterexample, and then discuss why the assertion appears to be so appealing.

Example: Let there be $N = 2$ potential bidders for a common-value asset: $v(T, X_i) = T$. Denote $\overline{M} = (\overline{m}, 0, 0)$, a second-price auction. Fix η; then $E[T]$ and $\mathcal{R}(\overline{M}, 2)$ are fixed. Then choosing information-acquisition cost $c = \left[E[T] - \mathcal{R}(\overline{M}, 2)\right]/2$ and bid preparation cost $b = 0$ yields an environment for which $\overline{M}$ is an optimal auction in the sequential entry model, if the equilibrium is selected in which the second potential bidder is indifferent over participating but participates with probability 1. The expected revenue attained is $\mathcal{R}(\overline{M}, 2)$. In the simultaneous entry model, $\pi(\overline{M}) = 1$ and $R(\overline{M}) = \mathcal{R}(\overline{M}, 2)$. However, by Proposition 2, an increase in φ from 0 to $d\varphi$ increases revenue, to a level unattainable in the sequential entry model.

The assertion pays attention to an obvious detriment in the main model, the probability $(1-\pi)^N$ that no potential bidder participates, and thus no gains from trade occur. It neglects a more subtle advantage: for optimal mechanisms, the probability that a participant faces a smaller-than-average number of rival participants is far larger than $(1-\pi)^N$. In the example, a potential bidder making a sequential participation decision knows for sure that he faces one rival bidder, and is indifferent over participating when $\varphi = 0$; an entry fee of $d\varphi > 0$ will lead to his nonparticipation and a plunge in revenue (to $d\varphi$). However, a potential bidder making a simultaneous participation decision will face one rival bidder with probability π^* slightly less than 1. If he faces one rival bidder, his net expected profitability is $-d\varphi$. Countering this loss is the $(1-\pi^*)$ probability that he faces no opposition and obtains the asset for a price of $d\varphi$. The seller gains because the resource costs have been reduced from $2c$ to $2\pi^* c$, and in each case there is a participant who is indifferent.

For those auction forms where the relationship between a bidder's expected profitability and an exogenously specified number of bidders is known, this relationship is strictly convex. Hence, a seller can sometimes attain a sizable π, even though the mechanism is strongly surplus-extractive, because a bidder is weighing in the chances of being the only participant or one of very few participants. With "sequential symmetric" entry, an optimal auction never faces a participant with fewer than $n_e(M) - 1$ rival participants. On average, the seller may be able to gain from this difference.

12 Appendix C: Proofs

Proof of Theorem 2: For the proof, shorten $\pi(M)$ to π, $\alpha(M, \pi, n)$ to α, $\mu_a(M, \pi, n)$ to μ_a, $\lambda_a(M, \pi, n)$ to λ_a, $\beta(n, N, \pi)$ to β_n, and $\beta(n-1, N-1, \pi)$ to β_{n-1}. Begin by harmlessly conditioning the price on the common trend, and then adding 0 in useful forms at two locations in

(2):

$$\begin{aligned}
& R(M) = (s_r E_\eta \{E_\eta [p(M,a,\pi,n,\cdot)|T]\} + a\varphi) \\
= & \sum_n \left\{ \sum_a (s_r E_\eta \{E_\eta [p(\cdot) - V(\cdot) - ab + V(\cdot) + ab|T]\} + a\varphi) \binom{n}{a} \mu_a - cn + cn \right\} \beta_n \\
= & \sum_n \left\{ s_r E_\eta [V(\cdot)] - b \sum_a a \binom{n}{a} \mu_a (M,\pi,n) - cn \right\} \beta_n \\
& + \sum_n \left\{ \sum_a (s_r E_\eta \{E_\eta [p(\cdot) - V(\cdot)|T]\} + a[\varphi + b]) \binom{n}{a} \mu_a + cn \right\} \beta_n \\
= & \sum_n \left\{ s_r E_\eta [V(\cdot)] - b \sum_a a \binom{n}{a} \mu_a (M,\pi,n) - cn \right\} \beta_n \\
& + \sum_n \left\{ \sum_a (s_r E_\eta \{E_\eta [p(\cdot)|T] - V(\cdot)\} + a[\varphi + b]) \frac{n}{a} \binom{n-1}{a-1} \lambda_a \alpha + cn \right\} \beta_n,
\end{aligned}$$

where the last equality uses the Bayes' formulas [(7) or (8)].

$$\begin{aligned}
& R(M) \\
= & \sum_n \left\{ s_r E_\eta [V(\cdot)] - b \sum_a a \binom{n}{a} \mu_a (M,\pi,n) - cn \right\} \beta_n \\
& + \sum_n \left\{ \sum_a (s_r E_\eta \{E_\eta [p(\cdot)|T] - V(\cdot)\} + [\varphi + b]) \binom{n-1}{a-1} \lambda_a \alpha + c \right\} n\beta_n \frac{\sum_i i\beta(i,N,\pi)}{\sum_i i\beta(i,N,\pi)} \\
= & \overline{T}(M) - b\overline{a}(M) - c\overline{n}(M) \\
& + \left[\sum_n \left\{ \alpha \sum_a (s_r E_\eta \{E_\eta [p(\cdot)|T] - V(\cdot)\} + [\varphi + b]) \binom{n-1}{a-1} \lambda_a + c \right\} \beta_{n-1} \right] N\pi,
\end{aligned}$$

where the first equality sorts n out of $\sum_a$ and multiplies by 1 in a useful form, and the final equality simplifies the numerator and combines the denominator with $n\beta_n$. The term in large $[\cdot]$ is 0 by (13). ∎

Proof of Theorem 3: Setting $\varphi = \sup_{\mathsf{B}} v(T,X)$ generates $\pi = 0$; $\varphi = -c - b$ generates $\pi = 1$. Interim expected profitability $\xi\left(\overline{M_\varphi}, n, \psi, x\right)$ is degenerate in φ, so the mapping $\varphi \mapsto \Xi^K\left(\overline{M}_\varphi, n, \varphi\right)$ [(6)] is continuous. Smoothness of $\mathcal{B}_1$ (from $\mathcal{A}$.1) implies that $\alpha^K\left(\overline{M}_\varphi, n\right)$ and $\lambda^K\left(\overline{M}_\varphi, a, n\right)$ are continuous in φ. Since, for $\overline{m}$, the price function $p(\cdot)$ is degenerate in φ, it follows that the $\pi\left(\overline{M}_\varphi\right)$ function implicitly defined in (13) is continuous in φ. The Intermediate Value Theorem yields the conclusion.

Proof of Corollary 2: Consider any $M = (m, \varphi, r)$ such that $\pi(M) = \widehat{\pi} \in (0,1)$, with r

nontrivial. By Theorem 3, there exists $\overline{M}_{\widehat{\varphi}} = (\overline{m}, \widehat{\varphi}, 0)$ so that $\pi\left(\overline{M}_{\widehat{\varphi}}\right) = \widehat{\pi}$. Expected revenue, $R(M)$, with r nontrivial, is

$$\begin{aligned} & \sum_n \left[\sum_a \left(E_\eta \left\{ V\left[M, \pi(M), n, \varphi, T\right] \right\} - ab \right) \left\{ s\left(M, a, \widehat{\pi}, n\right) \binom{n}{a} \mu_a\left(M, \pi, n\right) \right\} \right] \beta_n - c\widehat{\pi}N \\ < \; & \sum_n \left[\sum_a \left(E_\eta \left\{ V\left[M, \pi(M), n, \varphi, T\right] \right\} - ab \right) \left\{ s\left(\overline{M}_{\widehat{\varphi}}, a, \widehat{\pi}, n\right) \binom{n}{a} \mu_a\left(\overline{M}_{\widehat{\varphi}}, \pi, n\right) \right\} \right] \beta_n - c\widehat{\pi}N \end{aligned} \tag{16}$$

which is $R\left(\overline{M}_{\widehat{\varphi}}\right)$. Naturally, the probability that a given participant pays the reserve price is strictly less than the probability that he wins, while the probability that he pays the entry fee is strictly greater than the probability that he wins. Hence, with both mechanisms attaining participation probability $\widehat{\pi}$, $\widehat{\varphi} < \varphi + r$. The inequality then results from the terms in $\{\cdot\}$ on the left-hand side of (16) summing to less than the corresponding terms on the right-hand side (as terms corresponding to the cases where no participant is willing to pay the reserve price are zero on the left-hand side, but positive on the right, while the aggregate expected profit of participants is the same).

Proof of Proposition 1: [i]: The same π implies that (r.h.s.) of (13), which is monotone, attains the same value. Now reversing the substitutions used in the proof of Theorem 2 demonstrates revenue equality. [ii]: $\{\pi(M) = \pi(M')\} \Rightarrow \{\overline{n}(M) = \overline{n}(M')\}$, so [$ii$] follows from [$i$]. [$iii$]: The proof for the equality has already been shown. Suppose $\mathcal{R}(M, n) > \mathcal{R}(M', n)\ \forall n \in \mathbb{N}$. From (2), $\forall n \in \mathbb{N}$,

$$\begin{aligned} & \sum_a \left(E_\eta \left\{ E_\eta \left[p\left(M, \cdot\right) | T \right] \right\} + a\left[\varphi + b\right] \right) \binom{n}{a} \mu_a\left(M, \pi(M), n\right) \\ > \; & \sum_a \left(E_\eta \left\{ E_\eta \left[p\left(M', \cdot\right) | T \right] \right\} + a\left[\varphi' + b\right] \right) \binom{n}{a} \mu_a\left(M', \pi\left(M'\right), n\right) \end{aligned}$$

implies, using Bayes' formula as in the previous proof:

$$\sum_n \beta_{n-1} \left\{ \alpha\left(M', \pi', n\right) \sum_a \left[\frac{1}{a} E_\eta \left\{ T - E_\eta \left[p\left(M', \cdot\right) | T \right] \right\} - \varphi' - b \right] \binom{n-1}{a-1} \lambda_a\left(M', \pi\left(M'\right), n\right) \right\} > c,$$

implying $\pi(M) < \pi(M')$. The reverse inequality is identical.∎

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