GCOE Discussion Paper Series

Global COE Program
Human Behavior and Socioeconomic Dynamics

Discussion Paper No.206

Saving and Retirement Behavior under Quasi-Hyperbolic Discounting

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August 2011

GCOE Secretariat
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Abstract

This paper investigates saving and retirement behavior in a quasi-hyperbolic discounting model à la Laibson (1997) by incorporating endogenous labor supply. This behavior under quasi-hyperbolic discounting is characterized by: (i) comparing those with long-run optimal behavior, obtained under exponential discounting; and by (ii) comparing the behavior of sophisticated consumers with those of naïve consumers. Quasi-hyperbolic discounters, either naïve or sophisticated, definitely under-save and at the same time, if the wage rate is sufficiently low, retire earlier than long-run optimizers would. Consistent with empirical studies, therefore, under-saving and early-retirement can arise simultaneously.

Keywords: Saving, Retirement, Quasi-Hyperbolic discounting

* I am very grateful to Shinsuke Ikeda for his invaluable guidance through my research. I am thankful to Kouichi Futagami, Tatsuro Iwaisako and Yoshiyasu Ono for their comments and advice. I also thank to Shoko Morimoto for her helpful suggestions. All errors remain mine.
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1. Introduction

The quasi-hyperbolic discounting model is widely applied to study people's behavior of saving for retirement, for its good approximation to hyperbolic discounting's accurate description of time-inconsistent impatience – discounting the near future much more heavily than the distant future for the same length of time period, which is inspired by experimental research and common intuitions. On the other hand, time-consistent time preference can be described by exponential discounting under which the marginal rate of substitution of consumption between any two points of time depends only on how far apart the points are. Intuitively, the quasi-hyperbolic discounters do not save as much as the exponential discounters, driven by the stronger impatience to consume immediately. And as a result of this, they cannot accumulate enough wealth to support themselves after they stop working, have to delay the retirement age, and work for a longer time than the exponential discounters. This interplay between saving level and retirement age under quasi-hyperbolic discounting has been discussed by Laibson *et al.* (1998). On the other hand, retiring earlier is also desirable for the quasi-hyperbolic discounters; in order to realize it, they have to give up some current consumption and save at a higher level (Diamond and Koszegi (2003)). However, either under-saving or early-retirement can only be predicted separately by the previous studies. The aim of paper is to investigate that people could under-save and simultaneously early-retire under quasi-hyperbolic discounting.

Being widely observed, quasi-hyperbolic discounters' saving and retirement behavior is empirically investigated, even though they have not been jointly examined so far. Eisenhauer and Ventura (2006) find that wealth accumulation is negatively related to hyperbolic discounting in Italy and hyperbolic discounters are less likely to utilize commitment devices to control their choices. Fang and Silverman (2007) show that never-married mothers in USA fail to remain sufficient labor supply under quasi-hyperbolic discounting. By relating wealth level and labor supply to the discounting styles, under-saving jointly determined with early-retirement could be indirectly revealed. In this context, this theoretical analysis helps to explain this phenomenon which has not been jointly predicted by previous theoretical studies.

A series of empirical studies describe the indications of under-saving or early-retirement. Attanasio (1993) discovers that aggregate personal saving has declined in the USA since 1980s by accessing the CEX data set with the Bureau of Labor Statistic tapes of the Consumer Expenditure Survey. Bosworth *et al.* (1991) show that the American private saving rate has declined steadily over the last twenty years. Gendell (2001) shows that the average age at retirement declined in the 1990s. Gruber and Wise (2002) demonstrate that workers have been leaving the labor force at younger and younger age in recent years. These empirical studies jointly show that in the recent 20 years American household saving has tended to be less, and at the same time people have retired earlier.

The result of this paper could help to explain these empirical phenomena.

To demonstrate the results in this paper, a three-period quasi-hyperbolic discounting model incorporated with endogenous labor supply is employed. The saving level is determined in the first period and the retirement age in the second period. This setup is different from Diamond and Koszegi (2003) in that the labor supply chosen in the second period can be from 0 to 1, while it is "0 or 1" in Diamond and Koszegi (2003). Following O'Donoghue and Robin (1999), two types of consumers are distinguished: (i) people who can foresee their self-control problems are defined as sophisticated consumers; (ii) people who cannot recognize such problems are defined as naïve consumers. Furthermore, the instantaneous utility function is specified as the form of constant absolute risk aversion (CARA hereafter) in order to facilitate calculation, calibration and comparison of results.

I compare the saving and retirement behavior of different types of individuals in two aspects. Firstly, the saving level and retirement age under quasi-hyperbolic discounting are compared with those under exponential discounting. Following O'Donoghue and Robin (1999), the outcome of time-consistent preference is considered as best in the long run. In this context, the definition of under-saving and early-retirement is derived accordingly. Secondly, I compare the saving levels and retirement ages of the sophisticated consumers with those of the naïve consumers. By following Salanie and Treich (2006), I denote the present bias parameter separately according to different effects in each period and relate the present bias parameter in the second period to self-control problems.

The results found are as follows. (i) When the wage rate in the second period is sufficiently low, people with quasi-hyperbolic discounting, either sophisticated or naïve, will save less and jointly retire earlier than those with exponential discounting. This reveals the phenomenon of under-saving and early-retirement which is not predicted by the previous studies. (ii) Quasi-hyperbolic discounters definitely under-save. However, consistent with Diamond and Koszegi (2003), consumers under quasi-hyperbolic discounting may either early-retire or late-retire. (iii) Naïveté enhances under-saving and delays retirement age, and the difference in saving level and retirement age between the two types of discounters is shown to be monotonic in a present bias parameter. It is the present bias parameter in the second period which represents the self-control problem that differentiates the sophisticated consumers from the naïve ones.

The novelty of this paper is mainly proposed in the following aspects: the specification of constant absolute risk aversion instantaneous utility function is made to obtain closed-form solutions to the utility maximization problem with the present bias, and thereby conduct comparative dynamics with respect to the present bias parameter; and the difference in the behavior of the sophisticated consumers and the naïve consumers is proved to be monotonic in the present bias parameter. Thus, the present bias parameter is a key factor that differentiates not only quasi-hyperbolic discounters from exponential discounter but also sophisticated consumers from naïve consumers.

The remainder of this paper is organized as follows. Section 2 introduces the theoretical model and the solutions. Section3 is devoted to the results of naïve consumers' saving and retirement behavior. Section 4 discusses the solutions to the sophisticated consumers' problem. Section 5 compares sophisticated consumers with naïve consumers in their saving level and retirement age and investigates the effect of present bias parameter. Section 6 discusses the results got from the foregoing sections. Section 7 makes the concluding remarks.

2. The Model

To describe saving and retirement behavior of quasi-hyperbolic discounters, I extend the quasi-hyperbolic discounting model, à la Laibson (1997), by incorporating endogenous labor supply. The quasi-hyperbolic discounting model enables us to approximate the hyperbolic discounting preference in a tractable form.

Consider consumers, who live three periods, where the length of each period equals one. In the first period, people have no choice but to supply one unit of labor in-elastically at wage rate w_1, and at the same time, they decide the level of consumption c_1 and saving s_1.

Consumers get retired in the second period. Retirement decision is made by choosing period-2 working hours l between 0 and 1 for wage income $w_2 l$. Following Frogneux (2009), consumers have to endure the cost function of working e (l), which is assumed to be convex. After they retire in period 2, they do not receive income any more. In period 3, they just consume the amounts they saved before the retirement.

I follow Laibson (1997) in specifying the consumers' life time utility function with the present bias as the quasi-hyperbolic discounting model:

$$U_t = E_t\left[u(c_t) + \beta\sum_{\tau=1}^{3-t}\delta^{\tau}u(c_{t+\tau}) - (\beta\delta)^{2-t}e(l)\right], t = 1,2 \tag{1}$$

where 0< β <1 and 0< δ <1. I assume the utility function to be concave. The β captures the present bias whereas δ is representing the long-run discount factor. A smaller β implies a stronger present bias. When β equals 1, (1) reduces to the case of exponential discounting.

Specifically, from (1), the utility in period 1 is:

$$U_1 = u(c_1) + \beta\delta u(c_2) + \beta\delta^2 u(c_3) - \beta\delta e(l) \tag{2}$$

whereas the utility in period 2 is given by:

$$U_2 = u(c_2) + \beta\delta u(c_3) - e(l) \quad (3)$$

When present bias β is less than 1, the marginal rates of substitution between c_2 and c_3 in period 1 differs from that in period 2. It causes time-inconsistency if this problem is solved forwardly without incorporating the future shifting of the inter-temporal marginal rate of substitution. The consumers who do not notice this problem and solve the optimal problem forwardly are defined as being naïve. On the other hand, the consumers who regard the optimal problem as a game between selves in different periods and behave following the sub-game perfect Nash equilibrium solution by solving the problem backwardly are defined as being sophisticated. When there is no present bias (β =1) and hence no self-control problem, the solution of the naïve consumers coincides with that of the sophisticated consumers.

In order to simplify the calculation and thereby to give numerical examples later, I assume the instantaneous utility function is of constant absolute risk aversion (CARA) form.

Assumption 1 The instantaneous utility function in each period is:

$$u(c_t) = -e^{-Ac_t} \qquad (A>0) \quad (4)$$

This specification enables us to obtain closed-form solutions to the utility maximization problem with the present bias, and thereby conduct comparative dynamics with respect to β . Out of the same consideration, the assumption on the effort cost in the second period is made:

$$e(l) = e^{Bl} - 1 \qquad (B>0) \quad (5)$$

In what follows, the problem is solved in alternative cases to compare the solutions from two viewpoints. Firstly, I compare solutions under the present bias (β <1) and those under exponential discounting (β =1). As pointed out by O'Donoghue and Robin (1999), with quasi-hyperbolic discounting, solutions that would be obtained under exponential discounting gives those of the long-run optimizer who maximizes utility at the point in time just before he starts to consume. The comparison with solutions of exponential discounters is necessary to evaluate normatively the effect of the present bias. Specifically, I use the following terminology to characterize saving and retirement behavior normatively:

Definition: Consumers are said to *under-save* when their saving is smaller than that under exponential discounting. They are also said to get retired *early* when their retirement age is earlier than that under exponential discounting.

Secondly, I compare naïve consumers' solutions with sophisticated consumers'. By considering a single consumer as consisting of several inter-temporal selves in each period, these two types of consumers are included according to whether they are aware of their self-control problems. This enables me to consider the impacts of the level of sophistication.

3. Naïve Consumers

3.1 Solutions to naïve consumers problem

To start with, consider the case in which the consumers are naïve. By this assumption, the consumers cannot foresee their future self-control problem and the corresponding present-bias, they believe that they will carry out whatever plan they formulate today. I could derive their optimizing behavior by solving the utility maximization problems in periods 1 and 2 consecutively.

Period 1 Problem

Letting R be the gross rate of interest (R>1), the consumers' problem in period 1 is given by:

$$\underset{c_1,c_2,c_3,l}{Max}\, U_1 = u_1(c_1{}^N) + \beta\delta u_2(c_2{}^N) + \beta\delta^2 u_3(c_3{}^N) - \beta\delta e(l^N) \tag{6}$$

$$\text{s. t.}\quad c_3{}^N = R^2 w_1 + R w_2 l^N - R^2 c_1{}^N - R c_2{}^N \tag{7}$$

The first order condition is:

$$u_1'(c_1{}^N) = \beta\delta^2 R^2 u_3'(c_3{}^N) \tag{8}$$

$$u_2'(c_2{}^N) = \delta R u_3'(c_3{}^N) \tag{9}$$

$$e'(l^N) = \delta R w_2 u_3'(c_3{}^N) \tag{10}$$

Conditions (10) through (13) jointly determine the period 1 optimal solution for ($c_1{}^N$, $c_2{}^N$, $c_3{}^N$, and l^N). The consumers in period 1 actually consume the amount that equals the optimal solution for $c_1{}^N$, where it is obtained form (7) through (10).

$$c_1{}^N = \frac{ABR^2 w_1 - (2B + BR + ARw_2)\log(\delta R) - (B + BR + ARw_2)\log\beta}{A(B + BR + BR^2 + ARw_2)} \tag{11}$$

$$+ \frac{ARw_2 \log w_2 + ARw_2 \log(A/B)}{A(B + BR + BR^2 + ARw_2)}$$

$$s_1^N = w_1^N - c_1^N \tag{12}$$

$$= w_1 - \frac{ABR^2 w_1 - (2B + BR + ARw_2)\log(\delta R) - (B + BR + ARw_2)\log\beta}{A(B + BR + BR^2 + ARw_2)}$$

Although they also plan to carry out the period-1 solution for (c_2^N, c_3^N, and l^N), they will change the behavioral decisions in period 2, due to the present bias.

Period 2 Problem

Given the value of c_1^N and hence of savings $s_1^N = w_1 - c_1^N$, which were determined in period 1, the self in period 2 re-solves the optimization problem with respect to (c_2^N, c_3^N, and l^N).

$$\underset{l,c_2}{Max} U_2 = u_2(c_2^N) + \beta\delta u_3(c_3^N) - e(l^N) \tag{13}$$

$$\text{s. t. } \ c_3^N = R^2 s_1^N + Rw_2 l^N - Rc_2^N \tag{14}$$

The first order condition is:

$$u_2'(c_2^N) = \beta\delta R u_3'(c_3^N) \tag{15}$$

$$e'(l^N) = \beta w_2 \delta R u_3'(c_3^N) \tag{16}$$

The period-2 optimal solution for (c_2^N, c_3^N, and l^N) is determined by (14)-(16). In the presence of the present bias (β<1), note that the period-2 optimal (c_2^N,c_3^N, and l^N) differ from that of the period-1 optimal solution obtained form (7)-(10).

From (14) to (16) the solutions to the naïve consumers' problem under Assumption 1 are obtained as follows:

$$l^N = \frac{1}{(B + BR + ARw_2)(B + BR^2 + BR + ARw_2)}[-(B + BR + ARw_2)AR^2 w_1 \tag{17}$$

$$+ (B + BR + ARw_2 - BR^3 - AR^3 w_2)\log\beta + (B + BR + ARw_2)(1 + R^2 + R)\log w_2$$

$$+ (1 - R)(B + BR^2 + ARw_2)\log(\delta R) + (B + BR + ARw_2)(1 + R + R^2)\log((A/B)]$$

3.2 The effects of the present bias

The impacts of present-bias on saving and retirement behavior of the naïve consumers are investigated here. In order to examine how much the present-bias affects the saving level and retirement age, I differentiate(20) and (21) respect to β to obtain:

$$\frac{\partial c_1{}^N}{\partial \beta} = -\frac{B + BR + ARw_2}{A(B + BR + ARw_2 + BR^2)\beta} < 0 \tag{18}$$

$$\frac{\partial s_1{}^N}{\partial \beta} = -\frac{\partial c_1{}^N}{\partial \beta} = \frac{B + BR + ARw_2}{A(B + BR + ARw_2 + BR^2)\beta} > 0 \tag{19}$$

$$\frac{\partial l^N}{\partial \beta} = \frac{B + BR + ARw_2 - BR^3 - AR^3 w_2}{A(1 + R + Rw_2)(1 + R + Rw_2 + R^2)\beta} \tag{20}$$

Equation (23) implies that $\frac{\partial l^N}{\partial \beta} > 0$ if and only if $(B + BR + ARw_2 - BR^3 - AR^3 w_2) > 0$.

From the partial derivatives above, the naïve quasi-hyperbolic discounters always save less than the exponential discounters. Under the condition above, the retirement age of the naïve quasi-hyperbolic discounters is lower than that of the exponential discounters.

Proposition 1 Suppose that consumers are naïve. Then, in the setting that is specified in the previous section, a stronger present bias (i.e. a smaller β) leads to: (i) smaller saving definitely; and (ii) a earlier retirement age if and only if $w_2 < \frac{B}{A} \cdot \frac{1}{R(R+1)(R-1)} - 1$.

Proposition 1 implies that naïve consumers with β<1 definitely under-save in that their saving levels are definitely lower than the long-run optimal level, i.e. the saving level when β=1. Similarly, naïve consumers with β <1 get retired earlier that long-run optimizer would do when $w_2 < \frac{B}{A} \cdot \frac{1}{R(R+1)(R-1)} - 1$. w_2 is the marginal benefit got from working in period 2, while $\frac{B}{A} \cdot \frac{1}{R(R+1)(R-1)} - 1$ is regarded as the marginal cost. When the marginal benefit is sufficiently low, consumers will decide to retire early rather than late. Because even though they work longer, the income from working cannot cover their lives. In this circumstance of naïve consumers, under-saving and early-retirement arise, which is not consistent with the previous studies. In the later section I will show this is also the case when the consumers are sophisticated. These results are summarized as follows.

Corollary 1 Suppose that consumers are naïve. Then when $w_2 < \frac{B}{A} \cdot \frac{1}{R(R+1)(R-1)} - 1$, quasi-hyperbolic discounters under-save and jointly retire early.

It is straightforward to prove that there is a critical value $\overline{R}^N \in [1, +\infty)$ which satisfies $\frac{1}{\overline{R}^N (\overline{R}^N + 1)(\overline{R}^N - 1)} - 1 = 0$. Therefore, it is reasonable to conclude that the area of under-saving and early-retirement exists. See Figure 1 which demonstrates a special case of A=B. In other cases, the boundary line which separates early-retirement and late-retirement only deviates from the special case, but still guarantees the existence of early-retirement. The line in Figure 1 is the boundary between early-retirement and late-retirement. In the area below the boundary line early-retirement arises. The critical value $\overline{R}^N$ ensures the existence of this area.

4. Sophisticated Consumers

4.1 Solutions to sophisticated consumers problem

When people are sophisticated and incapable of commitment, their optimal behavior is obtained by solving backwardly the sub game-perfect equilibrium played by different inter-temporal selves.

Period 2 Problem

In period 3, the consumers just consume their wealth accumulated in periods 1 and 2. In period 2, they choose the consumption level in this period c_2 and decide when to retire (l) with a given saving level s_1 which is determined in period 1. The maximization problem faced by self 2 is:

$$\underset{l,c_2}{Max} U_2 = u_2({c_2}^S) + \beta\delta u_3({c_3}^S) - e(l^S) \tag{21}$$

$$\text{s. t.} \quad {c_3}^S = R^2 {s_1}^S + R w_2 l^S - R{c_2}^S \tag{22}$$

By solving the optimal problem, the first order conditions in period 2 are obtained as:

$$u_2'({c_2}^S) = \beta\delta R u_3'({c_3}^S) \tag{23}$$

$$e'(l^S) = \beta w_2 \delta R u_3'(c_3^{\ S}) \tag{24}$$

Period 1 Problem

In the Period 1, the consumption level c_1 is determined. The maximization problem faced by self 1 is:

$$\underset{c_1}{Max} U_1 = u_1(c_1^{\ S}) + \beta\delta u_2(c_2^{\ S}(s_1^{\ S})) + \beta\delta^2 u_3(c_3^{\ S}(s_1^{\ S})) - \beta\delta e(l^S(s_1^{\ S})) \tag{25}$$

$$\text{s. t. } \ c_3^{\ S}(s_1^{\ S}) = R^2 w_1 + Rw_2 l(s_1^{\ S}) - R^2 c_1^{\ S} - Rc_2^{\ S}(s_1^{\ S}) \tag{26}$$

And the first order condition in the first period is:

$$u_1(c_1^{\ S})' = \beta\delta^2 u_3(c_3^{\ S})'[-\frac{\partial c_3^{\ S}}{\partial c_1^{\ S}} - \beta R(\frac{\partial c_2^{\ S}}{\partial c_1^{\ S}} - w_2\frac{\partial l^S}{\partial c_1^{\ S}})] \tag{27}$$

The solutions to the sophisticated consumers' problem under Assumption 1are given as follows:

$$\begin{aligned} c_1^{\ S} = & \frac{1}{A(B+BR^2+BR+ARw_2)}[ABR^2 w_1 + ARw_2 \log w_2 - B\log\beta \\ & -(2B+ARw_2+BR)\log(\delta R) + ARW_2\log(A/B) \\ & -(B+ARw_2+BR)\log(B+\beta_{SC}BR+\beta_{SC}Aw_2R) \\ & +(B+ARw_2+BR)\log(B+BR+Aw_2R)] \end{aligned} \tag{28}$$

$$\begin{aligned} s_1^{\ S} &= w_1 - c_1^{\ S} \\ &= w_1 - \frac{1}{A(B+BR^2+BR+ARw_2)}[ABR^2 w_1 ARw_2 \log w_2 - B\log\beta \\ & \quad -(2B+ARw_2+BR)\log(\delta R) + ARW_2\log(A/B) \\ & \quad -(B+ARw_2+BR)\log(B+\beta BR+\beta Aw_2R) \\ & \quad +(B+ARw_2+BR)\log(B+BR+Aw_2R)] \end{aligned} \tag{29}$$

$$\begin{aligned} l^S = & \frac{1}{(B+BR^2+BR+ARw_2)}[(1+R_2+R)\log w_2 + \log\beta \\ & +(1-R^2)\log(\delta R) - AR^2 w_1 + (1+R+R^2)\log(A/B) \\ & -R^2\log(B+\beta BR+\beta Aw_2R) + R^2\log(B+BR+ARw_2)] \end{aligned} \tag{30}$$

4.2 The effects of the present bias

Similarly, I differentiate saving level and retirement age with respect to the present bias parameter β to compare the behavior under quasi-hyperbolic discounting with those under exponential discounting.

$$\frac{\partial s_1{}^S}{\partial \beta}=\frac{B^2+2\beta B^2R+2\beta ABRw_2+2\beta ABR^2w_2+\beta B^2R^2+\beta A^2R^2w_2{}^2}{\beta A[B+\beta BR+\beta ARw_2)][B+BR+ARw_2+BR^2]}>0 \qquad (31)$$

$$\frac{\partial l^S}{\partial \beta}=\frac{B+\beta BR+\beta ARw_2-\beta BR^3-\beta AR^3w_2}{A\beta(1+R+Rw_2)(1+R+Rw_2+R^2)} \qquad (32)$$

The derivative of saving level respect to the present bias parameter β is definitely positive. It implies that the saving level of the sophisticated quasi-hyperbolic discounters, whose present bias parameter β is less than 1, is definitely lower than that of the exponential discounters. Meanwhile, the derivative of retirement age respect to β can be either positive or negative.

Equation (48) implies that the necessary and sufficient condition of $\frac{\partial l^S}{\partial \beta}>0$ is $(B+\beta BR+\beta ARw_2-\beta BR^3-\beta AR^3w_2)>0$, which is similar to the condition of the naïve consumers. When β is close to 1 enough, this condition is the same as the one in the naïve consumers case. Therefore, if and only if $w_2<\frac{B}{A}\cdot[\frac{1}{\beta R(R+1)(R-1)}-1]$, the sophisticated quasi-hyperbolic discounter retire earlier than the exponential discounters do.

Proposition 2 Suppose that consumers are sophisticated. Then, in the setting that is specified in the previous section, a stronger present bias (i.e., a smaller β) leads to: (i) less saving definitely; and (ii) a earlier retirement if and only if $w_2<\frac{B}{A}\cdot[\frac{1}{\beta R(R+1)(R-1)}-1]$.

It is necessary to discuss whether the condition for $\frac{\partial l^S}{\partial \beta}>0$ really exists. Since $w_2>0$, the right hand side of this condition is supposed to be positive. After calculation, it can be obtained that for R>1, there exists a critical value $\overline{R}^S\in[1,+\infty)$ which makes $\frac{1}{\beta\overline{R}^S(\overline{R}^S+1)(\overline{R}^S-1)}-1=0$ and the condition for $\frac{\partial l^S}{\partial \beta}>0$ can be satisfied for $R\in[1,\overline{R}^S]$. We can also notice that $\overline{R}^S$ is the decreasing function of β ,i.e., a smaller β leads to a larger $\overline{R}^{S}$. And it is easier to meet the condition for $\frac{\partial l^{S}}{\partial \beta}$>0. To some extent, consumers with a higher level of present bias will retire more easily. This phenomenon could be obtained by comparing Figure 2 which demonstrates a special case of A=B. Otherwise, the boundary line which separates early-retirement and late-retirement only rotates around the special case, but still guarantees the existence of early-retirement.

So far from the general and specified aspect, I have formed the view that present bias can both lead to early-retirement and late-retirement.

Integrating the results above, if and only if $w_2 < \frac{B}{A} \cdot [\frac{1}{\beta R(R+1)(R-1)} - 1]$, the sophisticated quasi-hyperbolic discounters both under-save and retire early.

Corollary 2. Suppose that consumers are sophisticated. Then, quasi-hyperbolic discounters under-save and jointly retire early when $w_2 < \frac{B}{A} \cdot [\frac{1}{\beta R(R+1)(R-1)} - 1]$.

When the marginal benefit of working in period 2 is sufficiently low, people choose to retire early even though they under-save in period 1. This phenomenon is verified here in the case of the sophisticated consumers.

5. A static comparison: sophisticated consumers versus naïve consumers

The foregoing analysis investigates the saving level and retirement age of quasi-hyperbolic discounters in comparison with the exponential discounters. And then the following comparison I will make is between sophisticated consumers and naïve consumers. As naïve consumers, they cannot foresee their future self-control problem and corresponding present bias just as the results have been obtained above.

The consumption levels could be compared by subtraction:

$$c_1{}^{S} - c_1{}^{N} = \frac{1}{A(B+BR^2+BR+ARw_2)}[(BR+ARw_2)\log\beta + (B+ARw_2+BR)\log(B+BR+ARw_2) - (B+ARw_2+BR)\log(B+\beta BR+\beta ARw_2)] \quad (33)$$

Since $c_1^S - c_1^N |_{\beta \to 0} = -\infty$ and $c_1^S - c_1^N |_{\beta \to 1} = 0$; $c_1^S - c_1^N$ can be shown to be monotonic in β, which is proved in the Appendix. Therefore, $c_1^S < c_1^N$. On the other hand, the saving level of sophisticated consumers is higher than that of naïve consumers.

By the same way, the retirement age could be compared as follows:

$$\begin{aligned} l^S - l^N = & \frac{1}{(B+BR+ARw_2)(B+BR^2+BR+ARw_2)}[R^3(B+Aw_2)\log\beta \\ & + (B+ARw_2+BR)\log(B+BR+Aw_2R) \\ & - (B+ARw_2+BR)\log(B+\beta BR+\beta Aw_2R)] \end{aligned} \tag{34}$$

Since $l^S - l^N |_{\beta \to 0} = -\infty$, $l^S - l^N |_{\beta \to 1} = 0$ and $l^S - l^N$ is monotonic in β, $l^S < l^N$. See Figure 3. The retirement age of sophisticated consumers is lower than that of naïve consumers.

Even though either sophisticated consumers or naïve consumers behave under-saving, the saving levels of them diverge. In virtue of realizing their self-control problem, they make use of commitment devices to commit the saving behavior. Without the sophistication, the level of under-saving of naïve consumers is aggravated. Therefore, sophisticated consumers can work for a shorter time than the naïve one in the second period for they could enjoy the saving got from the first period.

Proposition 5 The saving level of the sophisticated consumers is higher than that of the naïve consumers. The retirement age of the sophisticated consumers is lower than that of the naïve consumers.

In order to consider the implications of the present bias parameter β, I follow Salanie and Treich (2006) in denoting β separately by its different effects in each period: the β that makes the discounting factor δ lower （$0 < \beta < 1$）in period 1, which is denoted by β_1; and the part of β that causes different marginal rate of substitution between c_2 and c_3, which is denoted as β_2. More explicitly, by using β_1 and β_2, the utility function in periods 1 and 2, (2) and (3), can be rewritten as

$$U_1 = u(c_1) + \beta_1 \delta u(c_2) + \beta_1 \delta^2 u(c_3) - \beta_1 \delta e(l), \tag{35}$$

$$U_2 = u(c_2) + \beta_2 \delta u(c_3) - e(l), \tag{36}$$

respectively. Therefore

$$c_1^S - c_1^N = \frac{1}{A(B+BR^2+BR+ARw_2)}[(BR+ARw_2)\log\beta_2 \quad (37)$$
$$+(B+ARw_2+BR)\log(B+BR+ARw_2)$$
$$-(B+ARw_2+BR)\log(B+\beta_2 BR+\beta ARw_2)]$$

$$l^S - l^N = \frac{1}{(B+BR+ARw_2)(B+BR^2+BR+ARw_2)}[R^3(B+Aw_2)\log\beta_2 \quad (38)$$
$$+(B+ARw_2+BR)\log(B+BR+Aw_2R)$$
$$-(B+ARw_2+BR)\log(B+\beta_2 BR+\beta Aw_2R)]$$

By denoting the present bias parameter β separately, the effects of β in differentiate the sophisticated consumers from the naïve consumers are clearly indicated. Both the differences of saving level and retirement age are only affected by β in period 2, i.e. β_2, but not β_1. Therefore, it is the present bias parameter in period 2 differentiates the sophisticated consumers from the naïve ones and causes the self-control problem.

Proposition 6 The divergence of saving level and retirement age between sophisticated consumers and naïve consumers is affected only by the present bias parameter in the second period.

6. Discussions

So far how the saving level and retirement age are affected by the discounting style has been investigated by taking partial of saving level and labor supply with respect to present bias parameter. And it is found that both the saving level and labor supply decrease with a decreasing present bias parameter β when the wage rate in the second period is sufficiently low, implying that quasi-hyperbolic discounters may under-save and jointly early-retire as contrasted with exponential discounters. In fact, quasi-hyperbolic discounters definitely under-save regardless of the condition. While the retirement behavior depends on the relationship of wage rate and gross interest rate. The wage rate w_2 is the marginal benefit of working in the second period, and the right side of the condition could be considered as the marginal cost of it. When the marginal benefit is lower than the marginal cost, it is reasonable for quasi-hyperbolic discounters to retire early in the second period since the longer they work the more they suffer from working.

Although this result is applicable to both sophisticated consumers and naïve consumers, the marginal conditions for early-retirement of them are not exactly the same. For naïve consumers, the condition for early-retirement is $w_2 < \frac{B}{A} \cdot \frac{1}{R(R+1)(R-1)} - 1$, while for sophisticated consumers it is $w_2 < \frac{B}{A} \cdot [\frac{1}{\beta R(R+1)(R-1)} - 1]$. The present bias parameter β differentiates the conditions. With exogenous wage rate w_2 and gross interest rate R, the condition for early-retirement of sophisticated consumers is easier to be achieved than that of naïve ones. In this context, sophisticated people are more like to retire early.

With the abnormity of under-saving and early-retirement, quasi-hyperbolic discounters suffer from double lost in the welfare than exponential discounters. Since a decreasing present bias parameter β leads to less saving and earlier retirement age, the quasi-hyperbolic discounters who are with smaller β suffer even more.

On the other hand, I find that in this model there is no evidence of over-saving which has been obtained by Diamond and Koszegi (2003). This is consistent with the result of Laibson (1997). However, the possibility of both early-retirement and late-retirement coincides with the results of Diamond and Koszegi (2003). In Laibson *et al.* (1998) as well as Diamond and Koszegi (2003), consumers have to face a tradeoff between consuming more and retiring early, although they are both desirable. In this paper, consumers also face this tradeoff, but when the wage rate of working is too low, even the income received from retiring later cannot support their lives.

A series of empirical studies describe the indication of under-saving or early-retirement. Attanasio (1993) discovers that aggregate personal saving declined in the USA since 1980s accessing to the CEX data set with the Bureau of Labor Statistic tapes of the consumer Expenditure Survey. Bosworth, Burtless, and Sablehaus (1991) show that the American private saving rate has declined steadily over the last twenty years. Gendell (2001) shows that the average age at retirement declined in the 1990s. Gruber and Wise (2002) demonstrate that workers are leaving the labor force at younger and younger age in recent years. These empirical studies jointly show that in the recent 20 years American household saving has tended to be less, and at the same time people got retired earlier. The result of this paper could help to explain these empirical phenomena.

During analyzing the quasi-hyperbolic discounters, two types of consumers are considered according to whether they are aware of their time-inconsistent preferences. With different level of sophistication, their saving level and retirement age diverge from each others'. It is showed that the saving level of naïve consumers is less than that of the sophisticated ones and the retirement age of naïve consumers is higher than that of the sophisticated ones. Naïveté enhances under-saving and therefore naïve consumers have to delay retirement age or else there will be no enough amounts to be consumed after stopping working.

By taking partially derivative, it is also found that the difference of consumption level and retirement age between sophisticated consumers and naïve consumers is monotonically increasing in the present bias parameter β. With the same level of present bias, the divergences of consumption level and retirement age between sophisticated consumers and naïve consumers are larger when present bias parameter β increases from 0 to 1. When β equals to 1, they converge to the exponential discounting case.

The separate denotation of present bias parameter allows for investigating the effect of them in each period. As Proposition 6 discusses, it is only the present bias in the second parameter that determine the difference of saving level and retirement age between sophisticated consumers and naïve consumers. The present bias parameter in the first period does not influence the divergence of consumers with different level of sophistication, only discounting the utilities in future.

7. Conclusion

In this study, saving and retirement behavior under quasi-hyperbolic discounting is investigated by comparing the case of quasi-hyperbolic discounting with that of exponential discounting. In the quasi-hyperbolic discounting case, both under-saving and early-retirement can exist simultaneously. Meanwhile, the behavior of the sophisticated consumers is compared with that of the naïve consumers. The self-control effect causes a deviation in the behavior of sophisticated consumers from that of the naïve consumers.

The wage rate and the gross interest rate appear to be the condition of early-retirement. However, they are exogenously determined in this paper. It would be interesting to extend this study in a general equilibrium model with endogenous wage rates and/or interest rates.

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Figure 1 **Naïve Consumers: the case of A=B**

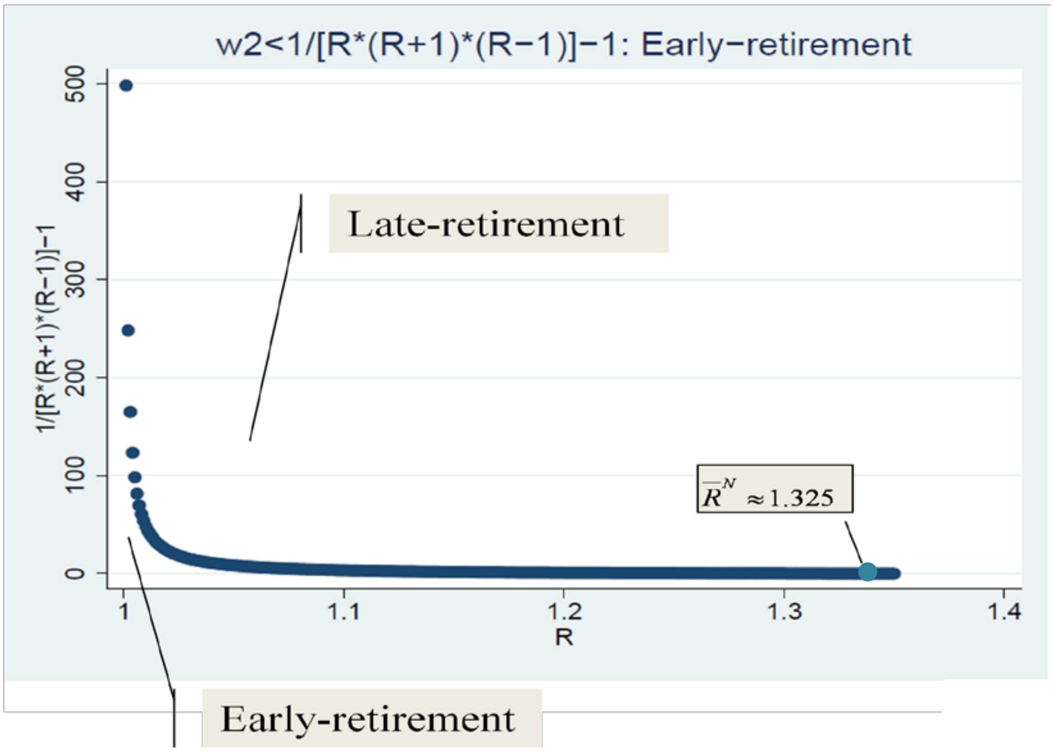


Figure 2 **Sophisticated Consumers: the case of A=B**

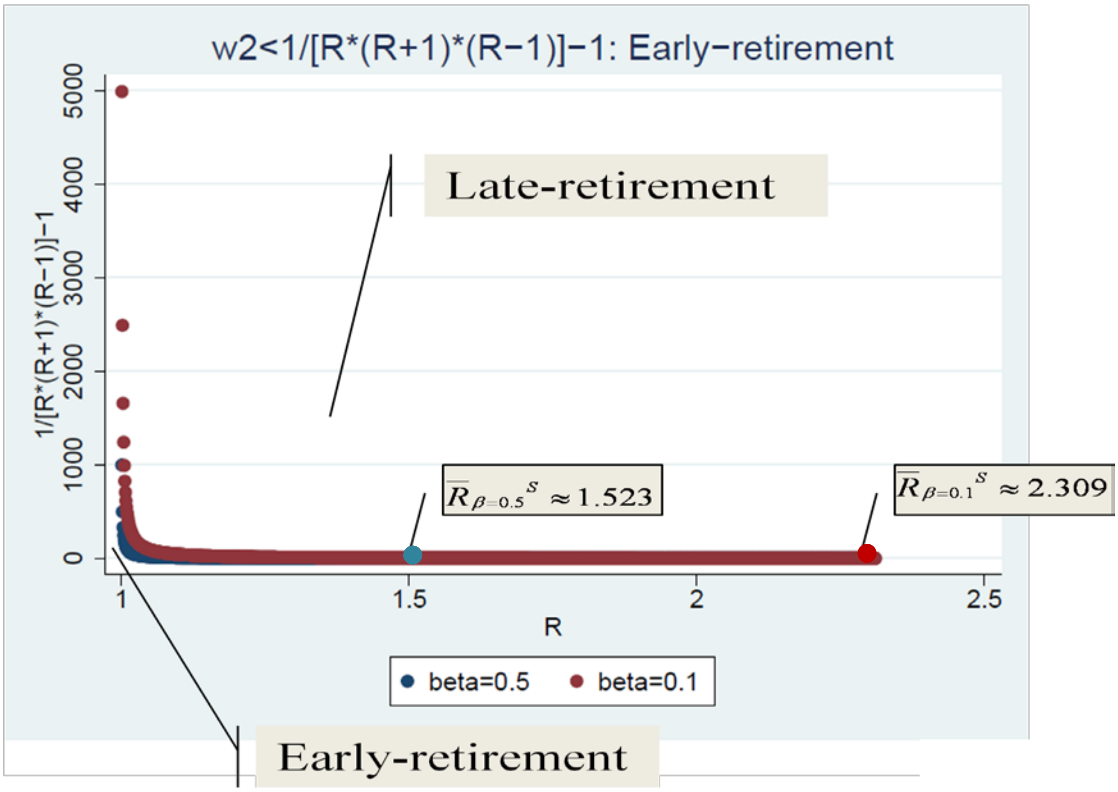
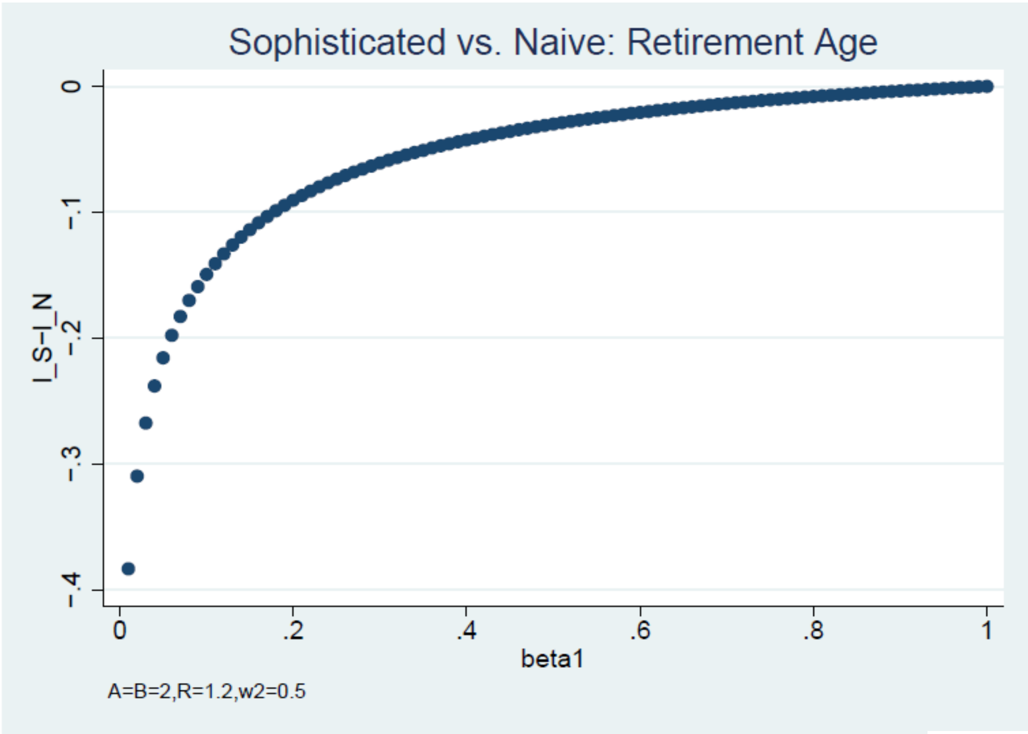


Figure 3


Sophisticated vs. Naive: Retirement Age
l_S−l_N
0
−.1
−.2
−.3
−.4
0
.2
.4
.6
.8
1
beta1
A=B=2,R=1.2,w2=0.5

Appendix

A.1 The monotonic property of $c_1^S - c_1^N$ and $l^S - l^N$

$$\frac{\partial(c_1^S - c_1^N)}{\partial\beta} = \frac{(1-\beta)BR(B+Aw_2)}{\beta(B+BR^2+ARw_2+BR)[B+\beta R(B+Aw_2)]} > 0$$

, so $c_1^S - c_1^N$ is monotonically increasing in β.

$$\frac{\partial(l^S - l^N)}{\partial\beta} = \frac{(1-\beta)BR(B+Aw_2)}{\beta(B+BR^2+ARw_2+BR)(B+ARw_2+BR)} > 0$$

, so $l^S - l^N$ is monotonically increasing in β.