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A Non-Unitary Discount Rate Model

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# A Non-Unitary Discount Rate Model *

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**Abstract**

The standard economic model of intertemporal decision making assumes that a single discount rate applies equally to discount (dis)utility from all different sources. However, studies such as psychology and behavioral economics have provided evidence that people might discount (dis)utility from different sources at different rates. This paper develops a simple model where the agent discounts utility from consumption at a different rate from disutility of labor supply. We show that in our non-unitary discount rate model, the preferences of the agent are time-inconsistent. The source of the time inconsistency is the difference between relative impatience with consumption and labor supply. It is shown that the policy effects in our model are quite different from those in the standard model. For example, when the agent discounts utility from consumption at a higher rate than the disutility of labor supply, the Friedman rule (the zero nominal interest rate) is no longer optimal. We also make comparisons between our results and those obtained in a model with a time variable discount rate where the preferences are time-inconsistent. It is also shown that the policy effects in our model are quite different from those in a model with a time variable discount rate.

**Keywords:** Non unitary discount rate, Tax policies, Time-inconsistency, Friedman rule.

**JEL Classification Numbers**:D9, E52, H21

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# 1 Introduction

Until the early 20th century, economists had been greatly concerned with various kinds of psychological and sociological motives that could determine intertemporal choices such as consumption and saving decisions. Intertemporal choices had been interpreted as the composite of many conflicting psychological and sociological motives, such as the bequest motive and temptations to consume too much today.[1] When in 1937 Samuelson proposed the discounted utility (henceforth, DU) model which was currently accepted as a standard model, however, many of the concerns about intertemporal choices that had been discussed until then were summarized by and compressed into a parameter, the discount rate.[2, 3] In the DU model proposed by Samuelson (1937), the intertemporal utility of an agent at time $t$ who lives to time $s(> t)$ without uncertainty is specified as $U_t = \int_t^s u(c_v)e^{-\rho(v-t)}dv$ where $u(c_v)$ is the instantaneous utility from time-$v$ consumption $c_v$ and $\rho$ is the subjective discount rate. We can extend the DU model to cases where an agent derives her utility from more than two different sources, like consumption and leisure. In such cases, the DU model assumes that a single discount rate is used commonly to discount (dis)utility from all different sources.

However, if each different source of (dis)utility is associated with a particular motive of intertemporal choice and hence people discount (dis)utility from different sources at different rates, the notion of a unitary discount rate is nonsense. Frederick et al. (2002) criticize the unitary discount rate assumption of the DU model, by arguing:

> When one looks at the behavior of a single individual across different domains, there is often a wide range of apparent attitudes toward the future. Someone may smoke heavily, but carefully study the returns of various retirement packages. Another may squirrel money away while, at the same time, giving little thought to electrical efficiency when purchasing an air conditioner. Someone else may devote two decades of his life to establishing a career, and then jeopardize this long term investment for some

[1]The early views of economists about intertemporal choices are well documented by Frederick et al. (2002).

[2]Frederick et al. (2002) provide an excellent review of the historical development of the DU model. The DU model has been widely accepted until now despite Samuelson's reservations about its validity.

[3]The other factors such as the curvature of the instantaneous utility function also affect intertemporal choices.

highly transient pleasure. (Frederick et al. (2002), p.393)

These behaviors of a single person cannot be explained if a single discount rate applies to discount (dis)utility from all different sources. A person who smokes heavily may discount the disutility of having poor health in the future at a higher rate. At the same time, her careful studying the returns of various retirement packages implies that she may discount utility from consumption after retirement at a much lower rate. In fact, there is evidence that people might discount (dis)utility from different sources at different rates. In Section 2, we present such evidence. Frederick et al. (2002) continue as follows:

> Since the DU model assumes a unitary discount rate that applies to all acts of consumption, such intra-individual heterogeneities pose a theoretical challenge. (Frederick et al. (2002), p.394)

Motivated by the above arguments, we present a simple model where a person discounts (dis)utility from different sources at different rates exponentially.

More precisely, we assume that the agent discounts utility from consumption at a different rate from the disutility of supplying labor. When the discount rate for utility from consumption is equal to that for the disutility of labor supply, our model reduces to a standard DU model. Therefore, we can easily compare the results obtained in our *non-unitary discount rate model*, where people use different discount rates to discount (dis)utility from different sources, with the results obtained in the standard DU model where people use a single discount rate to discount (dis)utility from all different sources.

We first show that in our non-unitary discount rate model, the marginal rate of substitution between consumption and labor supply is no longer time-invariant, and hence there emerges time inconsistency concerning the preferences of agents. When the agent discounts utility from consumption at a higher (lower) rate than the disutility of labor supply, she attempts to consume more (less) today and supply a larger (smaller) amount of labor today than she planned in the past.

Studies in behavioral economics suggest that the assumption of time consistency in the standard DU model is incorrect.[4] Authors such as Strotz (1955) and Laibson (1996,

[4]As to recent advances in behavioral economics, see Rabin (1998, 2002), Frederick et al. (2002) and

1997, 1998) show that the problem of time inconsistency emerges if individuals discount future utility with a time variable discount function, especially with the "quasi-hyperbolic" discount function.[5] In a model of hyperbolic discounting, as pointed out by O'Donoghue and Rabin (1999), the agent attempts to experience pleasant things immediately and to procrastinate regarding unpleasant things. This present-biased preference is the source of time inconsistency in a model of hyperbolic discounting.

In our non-unitary discount rate model, the difference between the patience with consumption and labor supply is the source of time inconsistency. Because our model assumes positive discount rates for both consumption and labor supply, the agent is willing to consume much today and to procrastinate regarding supplying labor today. If the discount rate for consumption is higher than for labor supply, however, the agent tends to be more willing to consume much today than to procrastinate regarding supplying labor today because she is relatively more impatient with decreases in consumption today than with increases in labor supply today. This difference in the patience is the source of time inconsistency in our model.

However, we do not claim that our model substitutes for models of hyperbolic discounting. The hyperbolic discount function is given by $v_p = V/(1 + kt)$ where $v_p$ is the present (discounted) value of an undiscounted value $V$, $t$ represents the time distance and $k(> 0)$ is a constant parameter representing the degree of discounting. As we will see in Section 2, some studies suggest that people use different hyperbolic discount functions (or different values of $k$) to discount (dis)utility from different sources. To isolate the roles of differences of discount functions from the roles of the hyperbolic discount function, we use exponential discount functions in this paper. If we use the hyperbolic discount function, our model corresponds to the case where people use different values of $k$ to discount (dis)utility from different sources.

To solve our non-unitary discount rate model formally, we consider the agent as composed of a sequence of autonomous decision makers as in many previous studies.[6] We call

Pesendorfer (2006), for example.

[5]The "quasi-hyperbolic" discount function used in Laibson (1996, 1997, 1998) and other studies is introduced by Phelps and Pollak (1968) in a model of imperfect intergenerational altruism.

[6]See Peleg and Yaari (1973), Goldman (1980), Harris and Laibson (2001) and Luttmer and Mariotti

the decision maker at time $t$ self $t$. Then we consider the choices of each decision maker (self) to be the outcome of an intrapersonal game. We show that in our non-unitary discount rate model, the consumption-saving behavior of the agent is affected by consumption taxes that have no effect on the consumption-saving behavior in the standard DU model.

To examine the welfare effects of taxes, we consider a simple general equilibrium where labor is used as the only input in production. We evaluate welfare from the perspective of all selves and derive policies that maximize the utility levels of all selves. In the standard DU model, the zero consumption tax rate is optimal. In our non-unitary discount rate model, however, it is shown that the utility levels of all selves can be improved by a strictly positive consumption tax (a consumption subsidy) when the agent discounts the utility from consumption at a higher (lower) rate than the disutility of labor supply. Furthermore, by introducing money under the assumption that a fraction of consumption goods must be financed by cash, we then show that when the agent discounts the utility from consumption at a higher rate than the disutility of labor supply, the Friedman rule is no longer optimal and development of the financial market (decreases in the fraction of consumption goods that must be financed by cash) deteriorates the utility levels of all selves.

Laibson (1996, 1997) also provides welfare implications similar to our results in a model with a time variable discount rate where the problem of time inconsistency exists. For example, Laibson (1997) shows that development of the financial market may deteriorate welfare. However, his analysis is based on a partial equilibrium model. To emphasize the importance of our results, we also conduct welfare analysis in a general equilibrium model where the agent uses a time variable discount function that is applied equally to consumption and labor supply. We show that in the general equilibrium model with a time variable discount rate, the zero consumption tax rate is optimal although the problem of time inconsistency exists. This result suggests that the strictly nonzero optimal consumption tax is not a common feature of general equilibrium models where the problem of the time inconsistency arises. Our results suggests that when the problem of time inconsistency exists in the economy, the optimal policy might be influenced by the sources of time

(2003), for example.

inconsistency.

The rest of the paper is organized as follows. Section 2 provides evidence that people might discount (dis)utility from different sources at different rates. In Section 3, we present our non-unitary discount rate model and show how the problem of time inconsistency emerges. Section 4 derives the solution of the intrapersonal game. By considering a simple general equilibrium model, Section 5 examines the effects of taxes on consumption-saving behavior and utility levels. Section 6 extends our model by introducing money. Section 7 compares the results obtained in our non-unitary discount rate model with those obtained in a model with a time variable discount rate. Concluding remarks are in Section 8.

## 2 Empirical and Experimental Evidence

As discussed in Introduction, the DU model assumes that a single discount rate applies equally to all types of goods and all categories of intertemporal choices. The studies such as psychology and behavioral economics report some empirical experimental observations that appear to contradict this assumption. Of such observations, we mention the *sign effect*, the *magnitude effect*, and the *domain effect* (or *domain independence*). What is most relevant to our model is the *domain effect*.

The *sign effect* refers to the finding that gains are discounted at a higher rate than losses. Loewenstein (1987) asked 30 undergraduates to determine how much you would pay most now to obtain (avoid losing) four dollars in the five different time delays. He found that on average, obtaining four dollars was discounted at higher rates than losing four dollars. Other authors, such as Thaler (1981), Benzion et al. (1989) and Abdellaoui et al. (2009) also found the sign effects.

Many studies have found that discount rates decrease with magnitudes of outcomes. More concretely, receiving \$1 million is discounted at lower rates than receiving \$100. This is often referred to as the *magnitude effect*. Many studies have found the magnitude effects.[7]

[7]see Thaler (1981), Benzion et al. (1989), Raineri and Rachlin (1993), Green, Fristoe, and Myerson

Many of previous studies were concerned with discount rates related to monetary outcomes. However, recent research has started to study discount rate of non-monetary outcomes. For example, Chapman and her co-authors have studied discount rates for money and health in a series of articles. Chapman (1996) conducted three experiments and found the low correlation between health and money discount rates, which suggested that a person who exhibited a high discount rate for money did not necessarily exhibit a high discount rate for health. She interpreted her result as showing that contrary to the DU model, people used the different discount rates for the two domains, money and health. Chapman and Elstein (1995) and Chapman et al. (1999) also reported the similar results. The finding that the discount rates differ for different domains is referred to as the *domain effect* (or *domain independence*). By using a sample of law students, Lazoro et al. (2001) found that the students did not apply the same discount rate for their choices about money and health. Baker et al. (2003) showed that both the current and never-before smokers discounted monetary loss at a higher rate than health losses.

The observations of the domain effects are not confined to money and health. Fuchs (1982) finds no correlations between a standard measure of time discounting ("Would you choose $1,500 now or $4,000 in five years?") and other behaviors that one might plausibly expect to be affected by time discounting (credit card debit, cigarette smoking, and the frequency of exercise and dental checkups). By using a sample of psychology students who had previous work experience and were seeking post-graduation jobs, Schoenfelder and Hantula (2003) found that students in their study used different discount rates to discount future salary outcomes and future access to attractive job duties. Loewenstein (1987) found that disutility from receiving electric shocks might be negatively discounted while receiving an amount of money was positively discounted.

Leclerc (1995) showed that money and time/effort were treated differently in decision making. The domain effect was observed for money and time/effort. Soman (1998) studied a monetary reward ($R$) and a loss of time/effort ($E$). In his experiments, subjects had to choose whether or not to enter a transaction where they would receive $R$ just after comple-

(1994), Myerson and Green (1995), Green, Myerson and McFadden (1997) and Kirby (1997), for example.

tion of $E$. Both $R$ and $E$ would occur at the same time in the future. Subjects who chose to enter this transaction might have evaluated the discounted value of $R$ much more than the discounted value of $E$. Soman (1998) observed that many of subjects who chose to enter the transaction did not actually redeem the required effort $E$ and could not get $R$. This suggested that after they had decided to enter the transaction, they might have changed their evaluation and then evaluated $R$ less than the costs of $E$.

If both $R$ and $E$ are discounted at the same rate (or by the same discount function), it is difficult to explain the above observation. Consider a person who discounts $R$ ($E$) by using a discount function $D_R(t) > 0$ ($D_E(t) > 0$).[8] Both $R$ and $E$ will occur after $t$ periods of time. If she evaluates $RD_R(t)$ more than $ED_E(t)$, which implies $RD_R(t) > ED_E(t)$, she chooses to enter the transaction. When $D_R(t)$ is equal to $D_E(t)$ for all $t \geq 0$, her decision to enter the transaction apparently implies $R > E$. This means that she actually redeems $E$ and can get $R$. If $D_R(t)$ is not equal to $D_E(t)$, however, the inequality $RD_R(t) > ED_E(t)$ does not necessarily imply $R > E$. Therefore, she might not redeem $E$. Soman (1998) interpreted his results as showing future time/effort was discounted at different speeds from future money. More specifically, the $k$ parameter of the hyperbolic discount function for effort was found to be different from that for money.[9] Soman (2004) and Zauberman and Lynch (2005) also showed that people used different discount rates to discount future time and future money.

The final evidence we provide suggests that people might use different discount rates to discount money- and labor-related (dis)utility. Table 1 is based on micro data from "Preference and Life Satisfaction Survey" (see Appendix A for details of this survey). Table 1 (a) shows that in the United States, of 6202 respondents in this survey who discount money-related utility at positive rates, about 70% of them (4317 respondents) use negative discount rates to discount the disutility of labor supply. Table 1 (b) provides similar results for Japan.

The above evidence raises doubts over the assumption of the DU model that a single discount rate applies equally to discount (dis)utility from all different sources. In the next section, we provide a model where agents use different discount rates to discount (dis)utility

[8] $D_R(t)$ ($D_E(t)$) is a decreasing function of $t$.

[9] Also, see Soman et al. (2005).

from different sources.

[Table 1]

# 3 The Model

We consider an infinitely-lived agent who is endowed with one unit of time that is allocated to labor or leisure at each moment of time. The preferences of the agent are given by:

$$U_t = \int_t^{+\infty} \left\{ u(c_v)e^{-\rho_c(v-t)} - v(l_v)e^{-\rho_l(v-t)} \right\} dv, \tag{1}$$

where $c_v \geq 0$ is the consumption level at time $v$ and $l_v \in [0, 1]$ is the time allocated to labor supply at time $v$. $u(c_v)$ and $v(l_v)$ represent the instantaneous utility derived from consumption and the instantaneous disutility of labor at time $v$, respectively. The functions, $u(\cdot)$ and $v(\cdot)$, are twice differentiable and satisfy $u'(\cdot) > 0$, $u''(\cdot) < 0$, $v'(\cdot) > 0$ and $v''(\cdot) > 0$. The parameters $\rho_c$ and $\rho_l$ are the subjective discount rates for consumption and labor supply, respectively. We assume $\rho_c > 0$ and $\rho_l > 0$ so that we obtain bounded utility (1), although some authors observe negative discount rates as discussed in Introduction. We allow the case where $\rho_c$ is not equal to $\rho_l$, which means that the agent discounts utility from different sources at different rates. When $\rho_c$ is (not) equal to $\rho_l$, we call a (non-)unitary discount rate case. When $\rho_c$ is larger (smaller) than $\rho_l$, if the importance that the agent puts on consumption at different times is compared with that of the disutility of labor at different times, the agent puts relatively greater (lesser) importance on consumption today than on future consumption, while the disutility from future labor supply is relatively more (less) important for her than the disutility from labor supply today. In other words, the agent is relatively more (less) impatient with decreases in consumption today than with increases in labor supply today.

As discussed in Sections 1 and 2, some studies suggest that people use different hyperbolic discount functions (or different $k$ parameters) to discount (dis)utility from different sources. However, it should be noted that we use exponential discount functions in (1) to isolate the effects of differences of discount functions from those of the hyperbolic discount function.

The budget constraint of the agent is given by:

$$\dot{a}_v = (1 - \tau^r) r_v a_v + (1 - \tau^w) w_v l_v - (1 + \tau^c) c_v + T_v, \tag{2}$$

where $a_v$ denotes the asset holdings at time $v$ and $r_v$ ($w_v$) is the interest rate (the wage rate). $\tau^r$, $\tau^w$, and $\tau^c$ are the interest income tax rate, the labor income tax rate, and the consumption tax rate, respectively. $\tau^r$, $\tau^w$ and $\tau^c$ are all assumed to be constant over time. The lump-sum transfer from the government is denoted by $T_v$. The budget of the government is balanced at any moment, $\tau^r r_v a_v + \tau^w w_v l_v + \tau^c c_v = T_v$.

## 3.1 Non-Unitary Discount Rate and Time Inconsistency

This subsection demonstrates that the problem of time inconsistency arises under preferences with non-unitary discount rates, by focusing on the case where $\tau^r$, $\tau^w$, $\tau^c$ and $T_v$ are all equal to zero. Before providing a formal solution in the next section, we consider the case where at time $t$, the agent chooses the sequence $\{c_v,\ l_v,\ a_v\}_{v=t}^{\infty}$ without considering the possibility that she reconsiders her choices at some future time. In other words, when she chooses the sequence $\{c_v,\ l_v,\ a_v\}_{v=t}^{\infty}$ at time $t$, she believes that at time $v(> t)$, she will obey the decision made at time $t$.

We maximize (1) subject to (2) by setting the present value Hamiltonian as follows:

$$H_v = u(c_v)e^{-\rho_c(v-t)} - v(l_v)e^{-\rho_l(v-t)} + \lambda_v(r_v a_v + w_v l_v - c_v),$$

where $\lambda_v$ is the costate variable associated with the asset holdings and $v$ is larger than $t$. From the first-order conditions, we obtain:

$$\frac{v'(l_v)}{u'(c_v)} e^{-(\rho_l - \rho_c)(v-t)} = w_v. \tag{3}$$

At time $t$, the agent plans to consume goods and supply labor according to (3) at time $v(> t)$.

If she maximizes her utility once again at time $v(> t)$, however, we obtain:

$$\frac{v'(l_v)}{u'(c_v)} = w_v. \tag{4}$$

In the unitary discount rate case ($\rho_c = \rho_l$), (3) is identical to (4). The decision made at time $v$ is consistent with that made at time $t(< v)$. In the non-unitary discount rate case ($\rho_c \neq \rho_l$), however, (3) is different from (4). The decisions at different dates are inconsistent. Note that as shown in the left-hand side of (3), in the non-unitary discount rate case, the marginal rate of substitution between consumption and labor supply is no longer time-invariant. The preferences of the agent are time-inconsistent.

[Figure 1]

Figure 1 shows graphically the time inconsistency of the non-unitary discount rate case. Figure 1 ignores asset holdings for expositional simplicity. In the next section, we present a formal solution of our model by considering asset holdings. The straight line represents the budget constraint, $c_v = w_v l_v$.[10] The curved lines are the indifference curves. Panel (a) shows the case where $\rho_c$ is larger than $\rho_l$. When the agent maximizes her utility at time $t$, the slope of the indifference curve for the time-$v(> t)$ instantaneous utility is given by $v'(l_v)e^{-(\rho_l-\rho_c)(v-t)}/u'(c_v)$. At point $A$, (3) holds. At time $t$, the agent plans to consume goods and supply labor at point $A$ in a future time $v(> t)$. When the agent maximizes her utility again at time $v$, however, the slope of the indifference curve of the time-$v$ instantaneous utility is given by $v'(l_v)/u'(c_v)$. At point $A$, $v'(l_v)/u'(c_v)$ becomes smaller than $v'(l_v)e^{-(\rho_l-\rho_c)(v-t)}/u'(c_v)$ because $\rho_c$ is larger than $\rho_l$. At time $v(> t)$, the agent wants to consume goods and supply labor at point $B$ where (4) holds, rather than to obey the plan made at time $t$ (point $A$). The agent likes to consume more and supply more labor at time $v$ than she planned in a past time $t(< v)$. The intuition is as follows: The inequality $\rho_c > \rho_l$ suggests that the agent is relatively more impatient with decreases in consumption today than increases in labor supply today. Therefore, the agent attempts to consume much today and

[10] Please note that we ignore asset holdings for expositional simplicity. Therefore, the budged constraint is given by $c_v = w_v l_v$.

cares less about the disutility of labor today. At each point of time, therefore, the agent attempts to consume more today and supply more labor today than she planned in the past.

Panel (b) in Figure 1 presents the case where $\rho_c$ is smaller than $\rho_l$. In this case, the agent is relatively more patient with decreases in consumption today than increases in labor supply today. The agent cares relatively less about decreases in consumption today and attempts to procrastinate about labor supply today. At each point of time, therefore, the agent attempts to consume less today and supply less labor today (point $D$) than she planned in the past (point $C$).

## 3.2 Comparison with a Time Variable Discount Rate Model

This subsection observes that the source of the time inconsistency in our non-unitary discount rate model is quite different from that of a model with a time variable discount rate. Consider the following utility function:

$$U_t = \int_t^{+\infty} \{u(c_v) - v(l_v)\} e^{-(\varrho\cdot(v-t)+\phi(v-t))} dv, \tag{5}$$

where $\varrho$ is a positive constant and $\phi(t)$ is a function of $t$. Following Barro (1999), we assume $\phi(0) = 0$, $\phi'(t) \geq 0$, $\phi''(t) \leq 0$ and $\lim_{t\to\infty} \phi'(t) = 0$. In (5), the instantaneous discount rate, $\varrho + \phi'(t)$, varies with time, and the same instantaneous discount rate applies equally to consumption and labor supply. If $v(l_v)$ is equal to zero for all $l_v \in [0, 1]$, (5) is equivalent to the utility function analyzed by Barro (1999). It is well known that when the discount rate is time variable as in (5), the preferences become time inconsistent. As pointed out by O'Donoghue and Rabin (1999), when the discount rate, $\varrho + \phi'(t)$, decreases with time, the preferences represented by (5) captures the tendency of the agent to attempt to experience pleasant things immediately and to procrastinate about unpleasant things. More precisely, at each moment of time, the agent endowed with (5) attempts to consume more today and enjoy more leisure today by procrastinating about labor supply than she planned in the past. This present-biased preference is the source of the time inconsistency in a model with a time variable discount rate.

In our non-unitary discount rate model, the difference between patience with consumption and labor supply is the source of time inconsistency. Remember that both $u(c)$ and $v(l)$ are positively discounted, $\rho_c > 0$ and $\rho_l > 0$. Therefore, the agent is willing to consume much today and to procrastinate about labor supply today. When $\rho_c > \rho_l$ holds, however, the agent tends to be more willing to consume much today than to procrastinate about labor supply today because she is more impatient with decreases in consumption today than with increases in labor supply today. This difference in the patience is the source of time inconsistency. Our non-unitary discount rate model may be appropriate for describing the situation where there are (more than) two distinct choice variables that the agent attempts to experience immediately (or procrastinate about), however, she is more willing to experience immediately (or procrastinate about) one of them than the other(s).

# 4 Generalized Euler Equation

This section provides a formal solution of our model by considering asset holdings. Following Peleg and Yaari (1973) and others, we consider the agent as composed of a sequence of autonomous decision makers who are indexed by time $t$. We call the decision maker at time $t$ self $t$. As in Pollak (1968) and others, we consider the choices of each self to be the outcome of an intrapersonal game. Following Barro (1999), we solve the intrapersonal game.

In the following analysis, we specify the instantaneous utility functions as:

$$u(c_v) = \frac{{c_v}^{1-\sigma}}{1-\sigma}, \quad \text{and} \quad v(l_v) = -\frac{\theta(1-l_v)^{1-\gamma}}{1-\gamma},$$

where neither $\sigma > 0$ nor $\gamma > 0$ are equal to one.[11] When $\sigma$ ($\gamma$) is equal to one, we assume the logarithmic utility function $u(c) = \log c$ ($v(l) = -\theta \log(1-l)$). A large $\theta(> 0)$ means that agents put relatively large weight on the disutility of labor supply. For analytical simplicity and to focus on the effects of the non-unitary discount rates, we consider the case where $\gamma$

[11] The disutility of labor is often specified as $v(l_v) = \theta {l_v}^{1+\gamma}/(1+\gamma)$ where $\gamma \geq 0$. If we use this specification in our model, however, it becomes difficult to obtain an analytical solution.

is equal to $\sigma$. In Appendix C, we examine the general case where $\gamma$ is different from $\sigma$.[12]

Given future selves' behaviors and the sequence of $\{r_v,\ w_v\}_{v=t}^{\infty}$, self $t$ chooses $c_t$ and $l_t$ that can be considered as constant flows over the infinitesimally short interval $[t, t+\Delta]$. The objective of self $t$ is then given by:

$$U_t = \int_t^{t+\Delta} z(v,t)dv + \int_{t+\Delta}^{\infty} z(v,t)dv \approx [u(c_v) - v(l_v)]\,\Delta + \int_{t+\Delta}^{\infty} z(v,t)dv, \tag{6}$$

where $z(v,t) \equiv u(c_v)e^{-\rho_c(v-t)} - v(l_v)e^{-\rho_l(v-t)}$. The approximation comes from setting $e^{-\rho_c(v-t)}$ and $e^{-\rho_l(v-t)}$ equal to one in the infinitesimal short interval $[t, t+\Delta]$.

Through the choice of $c_t$ and $l_t$, self $t$ can influence choices of selves $v(\geq\ t+\Delta)$ by affecting the asset holdings $a_{t+\Delta}$. To derive the optimal choices of self $t$, we first have to know the effects of $c_t$ and $l_t$ on $a_{t+\Delta}$, and second have to conjecture the policy functions of selves $v(\geq t+\Delta)$ to know the effects of $a_{t+\Delta}$ on future selves' choices.

The budget constraint (2) can be approximated as follows:

$$a_{t+\Delta} \approx \{1 + (1-\tau^r)r_t\Delta\}a_t + \{(1-\tau^w)w_t l_t - (1+\tau^c)c_t + T_t\}\Delta.$$

In this approximation, we ignore terms involving $\Delta^2$ and consider $r_t$ and $w_t$ to be constant in the infinitesimally short time interval $[t, t+\Delta]$. This equation implies that:

$$\frac{\partial a_{t+\Delta}}{\partial c_t} = -(1+\tau^c)\Delta, \quad \text{and} \quad \frac{\partial a_{t+\Delta}}{\partial l_t} = (1-\tau^w)w_t\Delta. \tag{7}$$

More consumption (labor supply) today leads to smaller (larger) asset holdings in the future.

We turn to the policy functions of self $v(\geq\ t+\Delta)$. We conjecture that self $v(\geq\ t+\Delta)$ chooses $c_v$ and $l_v$ so as to satisfy:

$$1 - l_v = (\theta\zeta_v)^{\frac{1}{\sigma}} c_v. \tag{8}$$

where we conjecture that $\zeta_v$ does not depend on the level of asset holdings. This conjecture

[12] As Appendix C shows, when $\gamma$ is not equal to $\sigma$, our non-unitary discount rate model cannot be solved without extreme assumptions.

turns out to be true. As in Barro (1999), we conjecture that the choices of self $t$ affect the levels of future consumption but not the shape of the path of future consumption. We conjecture that the path of future consumption is:

$$g_v^c \equiv \frac{\dot{c}_v}{c_v} = \frac{1}{\sigma}\left\{(1-\tau^r)r_v - \omega_v\right\}. \tag{9}$$

This specification allows $\omega_t$ to vary over time. We conjecture that $\omega_t$ does not depend on the level of initial assets. We will see that this conjecture also turns out to be true.

By integrating (2) from $t+\Delta$ to $+\infty$ and using (8) and (9), we obtain $(\mu_{t+\Delta}+\nu_{t+\Delta})c_{t+\Delta} = a_{t+\Delta} + W_{t+\Delta}$ where $W_t \equiv \int_t^\infty \{(1-\tau^w)w_v + T_v\}e^{-\int_t^v (1-\tau^r)r_s ds}dv$, $\mu_t \equiv \int_t^\infty (1+\tau^c)e^{\int_t^v \{g_s^c-(1-\tau^r)r_s\}ds}dv$ and $\nu_t \equiv \int_t^\infty (1-\tau^w)w_v(\theta\zeta_v)^{\frac{1}{\sigma}}e^{\int_t^v \{g_s^c-(1-\tau^r)r_s\}ds}dv$. Note that $a_{t+\Delta}$ has no effect on $\mu_{t+\Delta}$ and $\nu_{t+\Delta}$ because we conjecture that both $\zeta_v$ and $\omega_v$ do not depend on $a_{t+\Delta}$. We then have:

$$\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}} = \frac{1}{\mu_{t+\Delta}+\nu_{t+\Delta}}. \tag{10}$$

By using the policy functions of future selves, (8) and (9), we rewrite the objective function of self $t$, (6), as:

$$U_t = \left[\frac{{c_t}^{1-\sigma}}{1-\sigma} + \frac{\theta(1-l_t)^{1-\sigma}}{1-\sigma}\right]\Delta + \frac{{c_{t+\Delta}}^{1-\sigma}}{1-\sigma}e^{-\rho_c\Delta}\Phi_{t+\Delta} + \frac{\theta {c_{t+\Delta}}^{1-\sigma}}{1-\sigma}e^{-\rho_l\Delta}\Psi_{t+\Delta},$$

where $\Phi_t \equiv \int_t^\infty e^{\int_t^v \{(1-\sigma)g_u^c-\rho_c\}du}dv$ and $\Psi_t \equiv \int_t^\infty (\theta\zeta_v)^{\frac{1-\sigma}{\sigma}}e^{\int_t^v \{(1-\sigma)g_u^c-\rho_l\}du}dv$. Self $t$ chooses $c_t$ and $l_t$ so as to maximize this objective function. Note that $a_{t+\Delta}$ has no effects on $\Phi_{t+\Delta}$ and $\Psi_{t+\Delta}$ because we conjecture that both $\zeta_v$ and $\omega_v$ do not depend on $a_{t+\Delta}$. Then, the first-order conditions are given by:

$${c_t}^{-\sigma} = (1+\tau^c)X_{t+\Delta}\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}}, \quad \text{and} \quad \theta(1-l_t)^{-\sigma} = (1-\tau^w)w_t X_{t+\Delta}\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}},$$

where $X_{t+\Delta} \equiv c_{t+\Delta}^{-\sigma}e^{-\rho_c\Delta}\Phi_{t+\Delta} + \theta c_{t+\Delta}^{1-\sigma}e^{-\rho_l\Delta}\Psi_{t+\Delta}$. In deriving the first-order conditions, we use

(7). As $\Delta$ approaches zero, the first-order conditions become:

$$c_t^{-\sigma} = (1+\tau^c)X_t\frac{\partial c_t}{\partial a_t}, \quad \text{and} \quad \theta(1-l_t)^{-\sigma} = (1-\tau^w)w_t X_t\frac{\partial c_t}{\partial a_t}, \tag{11}$$

where $X_t = c_t^{-\sigma}\Phi_t + \theta c_t^{-\sigma}\Psi_t$.

From the two conditions of (11), together with (8), we obtain:

$$\zeta_t = \frac{(1+\tau^c)}{(1-\tau^w)w_t}. \tag{12}$$

Apparently, $\zeta_t$ does not depend on the level of asset holdings. Our conjecture turns out to be true.

The first condition of (11) and (10) implies:

$$\mu_t + \nu_t = (1+\tau^c)\left(\Phi_t + \theta\Psi_t\right). \tag{13}$$

This equation holds for all $t \geq 0$. We differentiate both sides with respect to $t$, and after some manipulations,[13] we obtain:

$$\omega_t = \frac{\rho_c\Phi_t + \rho_l\theta\Psi_t}{\Phi_t + \theta\Psi_t}, \tag{14}$$

where $\Phi_t \equiv \int_t^\infty e^{\int_t^v\{(1-\sigma)g_u^c - \rho_c\}du}dv$, $\Psi_t \equiv \int_t^\infty (\theta\zeta_v)^{\frac{1-\sigma}{\sigma}} e^{\int_t^v\{(1-\sigma)g_u^c - \rho_l\}du}dv$, $g_u^c \equiv \{(1-\tau^r)r_u - \omega_u\}/\sigma$ and $\zeta_v$ is given by (12). As we conjectured, $\omega_t$ does not depend on the level of asset holdings.

The behavior of the agent is summarized by:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma}\left\{(1-\tau^r)r_t - \frac{\rho_c\Phi_t + \rho_l\theta\Psi_t}{\Phi_t + \theta\Psi_t}\right\} \equiv g_t^c, \tag{15}$$

$$1 - l_t = \left\{\frac{\theta(1+\tau^c)}{(1-\tau^w)w_t}\right\}^{\frac{1}{\sigma}} c_t. \tag{16}$$

Also in the unitary discount rate case ($\rho_c = \rho_l = \rho$), the same equation as (16) is derived. We call (15) the generalized Euler equation. In the unitary discount rate case, (15) reduces

[13] In Appendix B, we present a derivation of (14).

to the standard Euler equation: $\dot{c}_t/c_t = \{(1-\tau^r)r_t - \rho\}/\sigma$.

In contrast, in the non-unitary discount rate cases ($\rho_c \neq \rho_l$), the generalized Euler equation takes a rather different form. In the log-utility case ($\sigma = 1$), however, the generalized Euler equation takes a simple form. When $\sigma$ is equal to one, we have $\Phi = 1/\rho_c$ and $\Psi = 1/\rho_l$ by definition. The generalized Euler equation reduces to:

$$\frac{\dot{c}_t}{c_t} = (1-\tau^r)r_t - \tilde{\rho},$$

where $\tilde{\rho} \equiv (1+\theta)\rho_c\rho_l/(\rho_l + \theta\rho_c)$. We can derive the same Euler equation by maximizing the following unitary discount rate utility function subject to (2):

$$U_t = \int_t^{\infty} (u(c_v) - v(l_v))e^{-\tilde{\rho}(v-t)}dv.$$

With logarithmic utility functions, the non-unitary discount rate model is observationally equivalent to a unitary discount rate model in which the discount rate is equal to $\tilde{\rho}$.[14] Furthermore, we obtain the following proposition.

**Proposition 1**

*Suppose that the instantaneous utility functions have logarithmic forms. Consider two agents, one of which has discount rates, $\rho_{c1}$ and $\rho_{l1}$. The other has $\rho_{c2}$ and $\rho_{l2}$. If $\rho_{c1}\rho_{l1}/(\rho_{l1} + \theta\rho_{c1}) = \rho_{c2}\rho_{l2}/(\rho_{l2} + \theta\rho_{c2})$ holds, the generalized Euler equations become the same for the two agents.*

When the utility functions are not logarithmic, the generalized Euler equation takes a more complex forms. Because the generalized Euler equation includes $w_v$, $\tau^c$ and $\tau^w$ ($v \geq t$) through $\Phi_t$ and $\Psi_t$, the consumption-saving behavior at time $t$ is influenced by $w_v$, $\tau^c$ and $\tau^w$. Remember that in our non-unitary discount rate model, the problem of time inconsistency arises. Given policy functions of the future selves, self today attempts to affect the future selves' behaviors in a preferable manner for self today by controlling the asset holdings left

[14]Using logarithmic utility, Pollak (1968), Barro (1999) and many others obtain similar observational equivalence in models with a time variable discount rate. Karp (2007) considers a more general utility function.

to the future selves. Therefore, the saving decision of self today is affected by behaviors of the future selves. Behaviors of the future selves are influenced by $w_v$, $\tau^c$ and $\tau^w$ ($v \geq t$). Consequently, the consumption-saving behavior of self today is influenced by $w_v$, $\tau^c$ and $\tau^w$. In the unitary discount rate case, the problem of time inconsistency does not arise. Therefore, $w_v$, $\tau^c$ and $\tau^w$ have no influence on the consumption-saving behavior of self today.

# 5 A Simple General Equilibrium Model

To examine the effects of preference parameters and taxes, we consider a simple general equilibrium model. Consider a competitive economy where there are identical firms. The number of firms is normalized to one. The representative firm produces a final good by using a constant-returns-to-scale technology, $Y_t = Al_t$, where $Y_t$ is the output level, $l_t$ is labor input and $A$ is a positive constant. Through profit maximization, the wage rate $w_t$ becomes equal to $A$.

The population size is normalized to one. We first consider an economy populated by homogeneous agents. Subsection 4.3 examines a case of heterogeneous agents. We assume that the initial asset holdings of the representative agent are zero, $a_0 = 0$. Because the agents are identical and there is no capital, $a_t$ is constant at zero over time. By using (16) and the goods market equilibrium condition, $c_t = Al_t$, we obtain:

$$c_E = \frac{A}{1 + A\left\{\frac{\theta(1+\tau^c)}{A(1-\tau^w)}\right\}^{\frac{1}{\sigma}}}, \quad \text{and} \quad l_E = \frac{1}{1 + A\left\{\frac{\theta(1+\tau^c)}{A(1-\tau^w)}\right\}^{\frac{1}{\sigma}}}. \tag{17}$$

Because $\sigma$ is strictly positive, we have:

$$\frac{\partial c_E}{\partial \tau^x} < 0, \quad \text{and} \quad \frac{\partial l_E}{\partial \tau^x} = \frac{1}{A}\frac{\partial c_E}{\partial \tau^x} < 0, \tag{18}$$

where $x = c$ or $w$. Because $c_E$ is constant, we have $\Phi_E = 1/\rho_c$ and $\Psi_E = \{\theta(1+\tau^c)/[A(1-\tau^w)]\}^{\frac{1-\sigma}{\sigma}}/\rho_l$.

From (15), we obtain the equilibrium interest rate:

$$r_E = \frac{1}{1-\tau^r} \frac{(1+\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}})\rho_c\rho_l}{\rho_l + \rho_c\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}}},$$

where $\zeta = (1+\tau^c)/\{A(1-\tau^w)\}$.

## 5.1 Consumption-Saving Behavior

By examining the effects on the equilibrium interest rate, we know the effects of the preference parameters and taxes on the consumption-saving behavior. In Figure 2, we depict a savings curve that represents the relationship between savings and the interest rate. The equilibrium interest rate is given by $r_E$. Suppose that changes in a parameter strengthen the saving incentives of each self. For any given interest rate, the savings of each self increases, which results in rightward shifts of the savings curve. The equilibrium interest rate must decrease from $r_E$ to $r'_E$. If a(n) decrease (increase) in the equilibrium interest rate is caused by changes in a parameter, therefore, we can conclude that changes in that parameter positively (negatively) affect the incentive to save.

[Figure 2]

The next proposition summarizes the effects of preference parameters on the equilibrium interest rate by assuming $\tau^c = \tau^w = \tau^r = 0$.

**Proposition 2**

(i) $\dfrac{\partial r_E}{\partial \rho_c} = \dfrac{{\rho_l}^2(1+\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}})}{(\rho_l+\rho_c\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}})^2} > 0,$ (ii) $\dfrac{\partial r_E}{\partial \rho_l} = \dfrac{{\rho_c}^2\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}}(1+\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}})}{(\rho_l+\rho_c\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}})^2} > 0,$

(iii) $\dfrac{\partial r_E}{\partial \theta} = \dfrac{\rho_c\rho_l(\theta\zeta)^{\frac{1-\sigma}{\sigma}}(\rho_l-\rho_c)}{\sigma(\rho_l+\rho_c\theta^{\frac{1}{\sigma}}\zeta^{\frac{1-\sigma}{\sigma}})^2} < (=)(>) 0$ if and only if $\rho_l < (=)(>) \rho_c$.

The first and second parts of Proposition 2 indicate that in an economy with relatively large discount rates, saving incentive are relatively weak.

The third part shows that in the unitary discount rate case, $\theta$ is irrelevant to the consumption-saving behavior. In the non-unitary discount rate cases, however, $\theta$ affects the consumption-saving behavior. In the case where $\rho_c$ is smaller than $\rho_l$, $\theta$ is negatively related to the incentive to save. An increase in $\theta$ indicates that the agent place a relatively large weight on the disutility from labor supply. Because $\rho_c$ is smaller than $\rho_l$, each self does not want to supply much labor today while she cares relatively less about the disutility from future labor supply. When $\theta$ increases, self today attempts to increase labor supply of future selves by reducing the asset holdings left to future selves. Consequently, savings decreases.

The next proposition examines the effects of tax rates and the wage rate ($w = A$).

**Proposition 3**

(i) $\dfrac{\partial r_E}{\partial \tau^r} > 0.$

(ii) $\dfrac{\partial r_E}{\partial \tau^x} < (=)(>)0, \quad$ if and only if $\quad (1-\sigma)(\rho_l - \rho_c) < (=)(>)0, \quad$ where $x = c$ or $w$.

(iii) $\dfrac{\partial r_E}{\partial A} < (=)(>)0, \quad$ if and only if $\quad (1-\sigma)(\rho_l - \rho_c) > (=)(<)0.$

(Proof) If we differentiate $r_E$ with respect to $\tau^r$, we obtain $\partial r_E/\partial \tau^r = r_E/(1-\tau^r) > 0$. We next differentiate $r_E$ with respect to $x$ where $x = \tau^c$, $\tau^w$ or $A$:

$$\frac{\partial r_E}{\partial x} = \frac{\rho_c \rho_l (\theta \zeta^{1-2\sigma})^{\frac{1}{\sigma}} (1-\sigma)(\rho_l - \rho_c)}{(1-\tau^r)(\rho_l + \rho_c \theta^{\frac{1}{\sigma}} \zeta^{\frac{1-\sigma}{\sigma}})^2} \frac{\partial \zeta}{\partial x}.$$

where $\partial \zeta/\partial \tau^c > 0$, $\partial \zeta/\partial \tau^w > 0$ and $\partial \zeta/\partial A < 0$. □

In both the unitary and the non-unitary discount rate cases, $\tau^r$ has the same qualitative effect on $r_E$. In contrast, $\tau^c$, $\tau^w$ and $w(= A)$ have different effects on $r_E$ in the two cases. While $\tau^c$, $\tau^w$ and $w(= A)$ have no effect in the unitary discount rate case, these three variables do influence the consumption-saving behavior in the non-unitary discount rate cases.

The intuition of the effects of $\tau^c$ is as follows. An increase in $\tau^c$ has two opposing effects. When $\rho_c$ is larger than $\rho_l$, each self attempts to consume much today, compared with future consumption. When $\tau^c$ increases, therefore, self today does not want to decrease consumption today while she cares relatively less about decreases in the future consumption levels. This negatively affects the saving incentives. However, because $\rho_c$ is larger than $\rho_l$, each self cares relatively less about the disutility of labor supply today, compared with the disutility of the future labor supply. Faced with an increase in $\tau^c$, self today attempts to decrease future labor supply more than labor supply today by saving more. This positively affects the saving incentives. When $\sigma$ is larger (smaller) than one, the negative (positive) effects dominate the positive (negative) effects. Consequently, the savings today decrease (increase). When $\rho_c$ is smaller than $\rho_l$, the opposite holds. When $\sigma$ is smaller (larger) than one, therefore, the savings today decrease (increase).

An increase in $w$ (a decrease in $\tau^w$) increases the incentive of labor supply, and has a positive effect on consumption. Therefore, an increase in $w$ (a decrease in $\tau^w$) has effects similar to a decrease in $\tau^c$. Then, we can obtain the results in Proposition 3.

## 5.2 The Welfare Effects of Taxes

We now examine the effects of taxes on welfare. Because the preferences of the agent are time-inconsistent, the different selves of an agent need not agree on their welfare ranking of the same consumption and labor supply sequences. In this paper, we evaluate welfare from the perspective of all selves following authors such as Laibson (1996, 1997).[15] In equilibrium, all selves have the same utility level which is given by:

$$U_E = u(c_E)/\rho_c - v(l_E)/\rho_l. \tag{19}$$

The interest income tax $\tau^r$ has no effects on utility. By using (17), we differentiate $U_E$ with respect to $\tau^x$ where $x = c$ or $w$:

$$\frac{\partial U_E}{\partial \tau^x} = c_E^{-\sigma}\left(\frac{1}{\rho_c} - \frac{1}{\rho_l}\frac{1-\tau^w}{1+\tau^c}\right)\frac{\partial c_E}{\partial \tau^x} < (>)0,$$

[15] As pointed out by O'Donoghue and Rabin (1999) and others, welfare comparisons for agents with time-inconsistent preferences are problematic because an agent's preferences at different times disagree. However, many studies, including Laibson (1996, 1997), Laibson et al. (1998) and İmrohoruğlu et al. (2003), often make welfare comparisons from the perspective of all selves.

if and only if $(1-\tau^w)/(1+\tau^c) < (>) \rho_l/\rho_c$ because $\partial c_E/\partial\tau^x$ has a negative sign (see (18)). We then obtain the next proposition.

**Proposition 4**

*The utility levels of all selves are maximized by setting* $\tau^c = \rho_c/\rho_l - 1$ $(\tau^w = 1-\rho_l/\rho_c)$ *when* $\tau^w$ $(\tau^c)$ *is equal to zero.*

Consider the effects of $\tau^c$. In the unitary discount rate case, the consumption tax (or subsidy) decreases the utility level. In contrast, in the non-unitary discount rate cases, the utility levels of all selves are improved by a consumption tax (subsidy), $\tau^c > 0$ $(\tau^c < 0)$, when $\rho_c$ is larger (smaller) than $\rho_l$. As discussed in Subsection 2.1, when $\rho_c$ is larger than $\rho_l$, self $v(>t)$ consumes more by supplying more labor than self $t$ prefers. Faced with a consumption tax, self $v(>t)$ reduces her own consumption and labor supply. Consequently, the consumption level and labor supply of self $v(>t)$ become close to those favorable for self $t$. Then, the utility level of self $t$ improves. Because all selves have the same utility level in equilibrium, the consumption tax can improve the utility of all selves.

Note that Proposition 4 holds even if the utility functions take logarithmic forms ($\sigma$ = 1). As shown in Proposition 1, when $\sigma = 1$ holds, the unitary discount rate economy becomes observationally equivalent to an economy with the non-unitary discount rate. In the economy with logarithmic utility, the government may misperceive the preferences of the agent. If the government believes that the agent is endowed with a unitary discount rate but the agent actually has the non-unitary discount rates, the government cannot implement policy in an appropriate manner because the effects of taxes on the welfare in these two cases are quite different, as shown in Proposition 4.

## 5.3 Heterogeneous Agents

This subsection briefly considers the case of heterogeneous agents, assuming logarithmic utility functions, $\sigma = 1$. The initial asset holdings of all agents are equal to zero. Let $\rho_c^i$ and $\rho_l^i$ be the subjective discount rates of agent $i$. We assume that $(1+\theta)\rho_c^i\rho_l^i/(\rho_l^i+\theta\rho_c^i) = (1+\theta)\rho_c^j\rho_l^j/(\rho_l^j+\theta\rho_c^j)(\equiv\tilde{\rho})$ holds for all $i$ and $j(\neq i)$ and that all agents have the same value of $\theta$. Because the behaviors of all agents are observationally equivalent (see Proposition 1) and because the initial asset holdings of all agents are equal to zero, it appears as if the economy is populated by identical agents. The equilibrium consumption level and labor supply of all agents are then given by (17). The equilibrium interest rate is $\tilde{\rho}$. The utility level of all selves of agent $i$ is: $U_E^i = u(c_E)/\rho_c^i - v(l_E)/\rho_l^i$. Assuming $\tau^w = 0$, we focus on the effects of $\tau^c$. Consider a small increase in $\tau^c$. As is clear from Proposition 4, the utility levels of all selves of agents with $\rho_c^i/\rho_l^i - 1 < (>)\tau^c$ decrease (improve). The utility levels of the agents with high $\rho_c^i/\rho_l^i$, who are relatively more impatient with decreases in consumption today, is improved by an increase in $\tau^c$.

# 6 An Extension: Monetary Economy

This section extends the basic model by introducing money. As in Section 4, the population size is normalized to one and we assume that the agents are identical. Subsection 5.1 considers the case of heterogeneous agents.

Let us denote the price level as $p_t$. We assume that a fraction of the purchase of consumption goods must be financed by cash. More precisely, to purchase $c_t dt$ units of consumption goods in a time interval of length $dt$, $\eta p_t c_t dt$ units of cash are needed in the same time interval. The parameter $\eta \in [0, 1]$ represents the fraction of consumption goods that must be purchased by cash. Let us denote the nominal cash holdings of agents at time $t$ as $M_t$. When an agent purchases $c_t$ units of consumption goods at time $t$, $M_t$ must satisfy $M_t \geq \eta p_t c_t$, or equivalently:

$$m_v \geq \eta c_v, \tag{20}$$

where $m_v \equiv M_v/p_v$. A larger $\eta$ means that agents need more cash for purchasing consumption goods. $\eta$ represents the degree of financial market development. As the financial market develops, $\eta$ decreases. The budget constraint is given by:

$$\dot{a}_v = r_v a_v - (r_v + \pi_v)m_v + w_v l_v - c_v + T_v, \tag{21}$$

where $\pi_v \equiv \dot{p}_v/p_v$ is the inflation rate, and $a_t$ is equal to $z_v + m_v$ where $z_v$ represents the asset holdings other than cash. We assume that at any moment of time, agents can allocate their portfolio between cash and other assets without any costs.

As for the money-supply behavior of the government, we assume a helicopter drop of money. The monetary authority issues nominal money at a positive and constant growth rate, $\epsilon \equiv \dot{M}_t/M_t$. The newly created money is transferred to agents as lump-sum payments. The budget constraint of the government is $p_t T_t = \epsilon M_t$.

As in Section 3, we solve the intrapersonal game. We begin with the effects on $a_{t+\Delta}$. The budged constraint (21) can be approximated as $a_{t+\Delta} \approx (1+r_t\Delta)a_t + \{w_t l_t - c_t - (r_t+\pi_t)m_t + T_t\}\Delta$ because we can ignore terms involving $\Delta^2$ and consider $r_t$, $w_t$, and $\pi_t$ to be constant in the infinitesimally short time interval $[t, t+\Delta]$. This equation implies that:

$$\frac{\partial a_{t+\Delta}}{\partial c_t} = -\Delta, \quad \frac{\partial a_{t+\Delta}}{\partial l_t} = w_t\Delta \quad \text{and} \quad \frac{\partial a_{t+\Delta}}{\partial m_t} = -(r_t + \pi_t)\Delta. \tag{22}$$

If self $t$ increases consumption or cash holdings (labor supply), the asset left to self $t+\Delta$ then decreases (increases).

We turn to the policy functions of self $v(\geq t+\Delta)$. As in Section 3, the choices of self $v(\geq t+\Delta)$ and the path of future consumption are conjectured as follows:

$$1 - l_v = (\theta\tilde{\zeta}_v)^{\frac{1}{\sigma}} c_v, \tag{23}$$

$$\tilde{g}_v^c \equiv \frac{\dot{c}_v}{c_v} = \frac{1}{\sigma}(r_v - \tilde{\omega}_v). \tag{24}$$

As in Section 3, we conjecture that $\tilde{\zeta}_v$ and $\tilde{\omega}_v$ do not depend on the level of asset holdings and that $\tilde{\zeta}_v$ and $\tilde{\omega}_v$ vary over time. In addition, we conjecture that self $v(\geq t+\Delta)$ does not hold more cash than needed for purchasing consumption goods, which means that (20) holds with equality for all $v(\geq t+\Delta)$. We will see that our conjectures turn out to be true.

From (20) with equality, (21), (23) and (24), we obtain $(\tilde{\mu}_{t+\Delta} + \tilde{\nu}_{t+\Delta})c_{t+\Delta} = a_{t+\Delta} + W_{t+\Delta}$ where $\tilde{\mu}_v \equiv \int_v^\infty \{1 + \eta(r_u + \pi_u)\} e^{\int_v^u (\tilde{g}_s^c - r_s) ds} du$ and $\tilde{\nu}_v \equiv \int_v^\infty w_u (\theta\tilde{\zeta}_u)^{\frac{1}{\sigma}} e^{\int_v^u (\tilde{g}_s^c - r_s) ds} du$. Note that $a_{t+\Delta}$ has no effects on $\tilde{\mu}_{t+\Delta}$ and $\tilde{\nu}_{t+\Delta}$ because we conjecture that both $\tilde{\zeta}_v$ and $\tilde{\omega}_v$ do not depend on $a_{t+\Delta}$. We then have:

$$\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}} = \frac{1}{\tilde{\mu}_{t+\Delta} + \tilde{\nu}_{t+\Delta}}. \tag{25}$$

The objective function of self $t$ is given by:

$$U_t = \left[\frac{{c_t}^{1-\sigma}}{1-\sigma} + \frac{\theta(1-l_t)^{1-\sigma}}{1-\sigma}\right]\Delta + \frac{{c_{t+\Delta}}^{1-\sigma}}{1-\sigma} e^{-\rho_c\Delta}\tilde{\Phi}_{t+\Delta} + \frac{\theta {c_{t+\Delta}}^{1-\sigma}}{1-\sigma} e^{-\rho_l\Delta}\tilde{\Psi}_{t+\Delta},$$

where $\tilde{\Phi}_t \equiv \int_t^\infty e^{\int_t^v \{(1-\sigma)\tilde{g}_u^c - \rho_c\} du} dv$ and $\tilde{\Psi}_t \equiv \int_t^\infty (\theta\tilde{\zeta}_v)^{\frac{1-\sigma}{\sigma}} e^{\int_t^v \{(1-\sigma)\tilde{g}_u^c - \rho_l\} du} dv$. Given the sequence of $\{r_v,\ w_v,\ p_v,\ \pi_v\}_{v=t}^\infty$, self $t$ maximizes this objective function subject to (22) and $m_t \geq \eta c_t$. We set the Lagrangian as follows: $\mathcal{L}_t = U_t + \lambda_t(m_t - \eta c_t)$ where $\lambda_t$ is the Lagrangian multiplier. Note that $a_{t+\Delta}$ has no effect on $\tilde{\Phi}_{t+\Delta}$ and $\tilde{\Psi}_{t+\Delta}$ because we conjecture that both $\tilde{\zeta}_v$ and $\tilde{\omega}_v$ do not depend on $a_{t+\Delta}$. Then, the first-order conditions are given by:

$$\left(c_t^{-\sigma} - c_{t+\Delta}^{-\sigma}\tilde{X}_{t+\Delta}\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}}\right)\Delta = \eta\lambda_t, \tag{26}$$

$$\theta(1-l_t)^{-\sigma} = c_{t+\Delta}^{-\sigma}\tilde{X}_{t+\Delta}\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}} w_t, \tag{27}$$

$$c_{t+\Delta}^{-\sigma}\tilde{X}_{t+\Delta}\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}}(r_t + \pi_t)\Delta = \lambda_t, \tag{28}$$

where $\tilde{X}_t = e^{-\rho_c\Delta}\tilde{\Phi}_t + \theta e^{-\rho_l\Delta}\tilde{\Psi}_t$.

The condition (28) implies $\lambda_t > 0$, which means that self $t$ does not holds more cash than needed for purchasing consumption goods, or equivalently (20) holds with equality for self $t$. Because this applies to self $v(\geq t+\Delta)$, our conjecture that (20) holds with equality for all $v(\geq t+\Delta)$ turns out to be true. From (26) and (28), we have:

$$c_t^{-\sigma} = c_{t+\Delta}^{-\sigma}\tilde{X}_{t+\Delta}\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}}\{1 + \eta(r_t + \pi_t)\}. \tag{29}$$

By using (23), (27), and (29), we obtain:

$$\tilde{\zeta}_t = \frac{1 + \eta(r_t + \pi_t)}{w_t}.$$

As $\Delta$ approaches zero, we have $\tilde{\mu}_t + \tilde{\nu}_t = \{1 + \eta(r_t + \pi_t)\}(\tilde{\Phi}_t + \theta\tilde{\Psi}_t)$ from (25) and (29). As in Section 3, by differentiating both sides of this equation with respect to time, we obtain:

$$\tilde{\omega}_t = \frac{\rho_c\tilde{\Phi}_t + \rho_l\theta\tilde{\Psi}_t}{\tilde{\Phi}_t + \theta\tilde{\Psi}_t} + \frac{\eta(\dot{r}_t + \dot{\pi}_t)}{1 + \eta(r_t + \pi_t)}.$$

As we conjectured, $\tilde{\zeta}$ and $\tilde{\omega}$ do not depend on the level of asset holdings.

For analytical simplicity, we proceed by assuming the logarithmic utility functions ($\sigma = 1$). If $\sigma$ is not equal to 1, we can obtain the same qualitative results. Because we have $\tilde{\Phi}_t = 1/\rho_c$ and $\tilde{\Psi}_t = 1/\rho_l$ when $\sigma = 1$ holds, the behavior of self $t$ is summarized by:

$$\frac{\dot{c}_t}{c_t} = r_t - \tilde{\rho} - \frac{\eta(\dot{r}_t + \dot{\pi}_t)}{1 + \eta(r_t + \pi_t)}, \tag{30}$$

$$1 - l_t = \frac{\theta\{1 + \eta(r_t + \pi_t)\}}{w_t}c_t, \tag{31}$$

where $\tilde{\rho} \equiv (1 + \theta)\rho_c\rho_l/(\rho_l + \theta\rho_c)$.

As in Section 4, we consider the simple general equilibrium. The production technology is $Y_t = Al_t$, where $Y_t$ is the output level, $l_t$ is labor input and $A$ is a positive constant. Through profit maximization, the wage rate $w_t$ becomes equal to $A$. Because there is no capital, we have $a_t = m_t$. We focus on the steady state equilibrium where $\dot{c}_t = \dot{r}_t = \dot{\pi}_t = 0$ holds. Equation (30) implies $r_t = \tilde{\rho}$. Because $m_t = \eta c_t$ implies $\dot{c}_t/c_t = \epsilon - \pi_t$, $\pi_t$ is equal to $\epsilon$. Because the nominal interest rate $\tilde{\rho} + \epsilon$ cannot be negative, $\epsilon$ must be equal to or larger than $-\tilde{\rho}$. By using (31) and $c_t = Al_t$, we obtain:

$$c^* = \frac{A}{1 + \theta\{1 + \eta(\tilde{\rho} + \epsilon)\}}, \quad \text{and} \quad 1 - l^* = \frac{\theta\{1 + \eta(\tilde{\rho} + \epsilon)\}}{1 + \theta\{1 + \eta(\tilde{\rho} + \epsilon)\}}. \tag{32}$$

Apparently, we have $\partial c^*/\partial x < 0$ and $\partial l^*/\partial x < 0$ where $x = \epsilon$ or $\eta$. The utility levels of all selves are given by $U^* = (\ln c^*)/\rho_c + \{\theta \ln(1 - l^*)\}/\rho_l$.

To derive $\epsilon$ that maximizes $U^*$ (henceforth, $\epsilon^*$), we differentiate $U^*$ with respect to $\epsilon$ by using (32):

$$\frac{\partial U^*}{\partial \epsilon} = \frac{\theta\eta}{1 + \theta\{1 + \eta(\tilde{\rho} + \epsilon)\}}\left\{\frac{1}{\rho_l\{1 + \eta(\tilde{\rho} + \epsilon)\}} - \frac{1}{\rho_c}\right\}. \tag{33}$$

By examining the sign of $\partial U^*/\partial\epsilon$, we obtain the next proposition.

**Proposition 5**

$$\epsilon^* = \begin{cases} -\tilde{\rho}, & \text{if} \quad \rho_c \le \rho_l, \\ \frac{1}{\eta}\left(\frac{\rho_c}{\rho_l} - 1\right) - \tilde{\rho}(> -\tilde{\rho}), & \text{if} \quad \rho_c > \rho_l, \end{cases}$$

Note that the real interest rate $r$ is equal to $\tilde{\rho}$. When $\epsilon^*$ is equal to $-\tilde{\rho}$, the nominal interest rate becomes equal to zero. In the unitary discount rate case and in the non-unitary discount rate case where $\rho_c$ is smaller than $\rho_l$, the Friedman rule is optimal. When $\rho_c$ is larger than $\rho_l$, $\epsilon^*$ is larger than $-\tilde{\rho}$. The Friedman rule is not optimal. When $\rho_c > \rho_l$ holds, self $v(> t)$ attempts to consume more by supplying more labor than self $t(< v)$ prefers. An increase in $\epsilon$ reduces the future selves' purchasing power, which causes decreases in $c^*$. Furthermore, $l^*$ also falls because of decreases in final goods production. When $\epsilon$ increases, consequently, consumption level and labor supply of self $v(> t)$ become close to those favorable for self $t$. Then, the utility level of all selves improves. When $\rho_c > \rho_l$ holds, therefore, the monetary authority can improve the utility levels of all selves by setting the nominal interest rate at a strictly positive level.

Because we consider the case where the unitary discount rate economy becomes observationally equivalent to an economy with the non-unitary discount rate, the monetary authority possibly misperceives the preferences of the agent. If the monetary authority believes that the agent is endowed with a unitary discount rate and if $\rho_c$ is actually larger than $\rho_l$, the monetary authority cannot implement policy in an appropriate manner.

We next examine the financial market development (decreases in $\eta$) by keeping $\epsilon$ constant at some level. Because $\eta$ disappears from $U^*$ when $\epsilon$ is equal to $-\tilde{\rho}$, we assume $\epsilon > -\tilde{\rho}$. Given $\epsilon(> -\tilde{\rho})$, we differentiate $U^*$ with respect to $\eta$ by using (32):

$$\frac{\partial U^*}{\partial \eta} = \frac{\theta(\tilde{\rho} + \epsilon)}{1 + \theta\{1 + \eta(\tilde{\rho} + \epsilon)\}} \left\{ \frac{1}{\rho_l\{1 + \eta(\tilde{\rho} + \epsilon)\}} - \frac{1}{\rho_c} \right\}. \tag{34}$$

When $\rho_c \le \rho_l$, $\partial U^*/\partial\eta$ has a negative sign. On the other hand, when $\rho_c > \rho_l$, $\partial U^*/\partial\eta$ has a positive (negative) sign if and only if $\eta < (>)(\rho_c - \rho_l)/\{\rho_l(\tilde{\rho} + \epsilon)\} \equiv \overline{\eta}$. Because decreases in $\eta$ represent financial market developments, we obtain the next proposition.

**Proposition 6**

*As the financial market develops,*

1. *when $\rho_c \leq \rho_l$ holds, the utility levels of all selves increase;*
2. *when $\rho_c > \rho_l$ holds, the utility levels of all selves increase if the financial market is less-developed ($\eta > \overline{\eta}$), while the utility levels of all selves decrease if the financial market is well-developed ($\eta < \overline{\eta}$).*

As $\eta$ decreases, the constraint on consumption purchases (20) becomes loose. This has a positive effect on $U^*$. Because there exists only this positive effect when $\rho_c \leq \rho_l$ holds, decreases in $\eta$ improve the utility levels of all selves. When $\rho_c > \rho_l$ holds, however, a negative effect is also at work. When $\rho_c > \rho_l$ holds, self $v(> t)$ attempts to consume more by supplying more labor than self $t(< v)$ prefers. As $\eta$ decreases, the future selves increase their consumption further, which results in increases in labor supply because of the rise in the final goods production. The differences between consumption levels (labor supplies) of the future selves and those favorable for self $t(< v)$ become wider. As a result, a decrease in $\eta$ negatively affects $U^*$. In an economy with a less-developed (well-developed) financial market, the positive (negative) effect dominates the negative (positive) effect. The financial market development improves (diminishes) the utility levels of all selves.

## 6.1 Heterogeneous Agents

As in Subsection 4.2.1, we assume that $(1 + \theta)\rho_c^i\rho_l^i/(\rho_l^i + \theta\rho_c^i) = (1 + \theta)\rho_c^j\rho_l^j/(\rho_l^j + \theta\rho_c^j)(\equiv \tilde{\rho})$ holds for all $i$ and $j(\neq i)$ where $\rho_c^i$ and $\rho_l^i$ are the subjective discount rates of agent $i$ and that the initial asset holdings of all agents are equal to zero. All agents have the same value of $\theta$ and the utility functions are logarithmic. Because the behaviors of all agents are observationally equivalent (see Proposition 1) and the initial asset holdings of all agents are equal to zero, it looks as if the economy is populated by identical agents. We focus on the steady state equilibrium. The equilibrium consumption level and labor supply of all agents are then given by (32). The equilibrium interest rate is $\tilde{\rho}$. The utility level of all selves of agent $i$ is given by $U^{i*} = u(c^*)/\rho_c^i - v(l^*)/\rho_l^i$. We can use Propositions 5 and 6 to evaluate welfare effects. When $\epsilon(\geq \tilde{\rho})$ increases, the utility levels of all selves of agents with $\rho_c^i \leq \rho_l^i$ decrease. The utility of all selves of agents with $\rho_c^i > \rho_l^i$ decreases (increases) if $\rho_c^i/\rho_l^i < (>)1 + \eta(\epsilon + \tilde{\rho})$ holds. As $\eta$ decreases, the utility levels of all selves with $\rho_c^i \leq \rho_l^i$ increase. The utility of all selves with $\rho_c^i > \rho_l^i$ decreases (increases) if $\eta < (>)\overline{\eta}^i \equiv (\rho_c^i - \rho_l^i)/\{\rho_l^i(\tilde{\rho} + \epsilon)\}$ holds. Note that when $\tilde{\rho}$ is kept constant, $\overline{\eta}^i$ increases with $\rho_c^i/\rho_l^i$. The utility levels of the agents with high $\rho_c^i/\rho_l^i$, who are relatively more impatient with decreases in consumption today, tend to be increased by increases in the inflation rate and to be decreased by the development of the financial market.

## 7 Optimal Policy and a Time Variable Discount Rate

Propositions 4, 5 and 6 provide important welfare implications. Laibson (1996, 1997) provides results similar to Propositions 4, 5 and 6. For example, Laibson (1997) shows that the development of the financial market may deteriorate welfare in a model with a time variable discount rate where the problem of time inconsistency arises. However, his analysis is based on a partial equilibrium model. To emphasize the importance of our results, we consider a general equilibrium model with a time variable discount rate that is similar to Barro (1999) by assuming that the agents are identical and the population size is one.

Instead of (1), this section assumes (5). Please note that even if the instantaneous utility functions have logarithmic forms, Propositions 4, 5 and 6 hold in the non-unitary discount rate model of the previous sections. For simplicity, we assume the logarithmic utility functions: $u(c) = \ln c$ and $v(l) = -\theta \ln(1 - l)$. When $\theta$ is equal to zero, (5) becomes exactly the same as the utility function employed in Barro (1999). The budget constraint is:

$$\dot{a}_v = r_v a_v + w_v l_v - (1 + \tau^c)c_v + T_v. \tag{35}$$

Because we are interested in the optimal consumption tax, the other taxes are omitted and money is excluded in this section.

It is well known that when the discount rate varies with time as in (5), the problem of time inconsistency arises. As in Section 3, we consider the agent as composed of a sequence of autonomous decision makers. If we follow the same procedure as in Section 3, we can derive the behavior of self $t$, which is summarized by:

$$\frac{\dot{c}_t}{c_t} = r_t - \xi, \tag{36}$$

$$1 - l_t = \theta \frac{1+\tau^c}{w_t} c_t, \tag{37}$$

where $\xi \equiv 1/\int_0^\infty \exp\{-(\varrho t + \phi(t))\}\, dt$.[16] Barro (1999) obtains the same Euler equation as (36). Equation (37) is the same as (16) (if we set $\tau^w = 0$ in (16)). Note that the model with the time variable discount rate is observationally equivalent to the non-unitary discount rate model if $\xi$ is equal to $\tilde{\rho}(\equiv (1+\theta)\rho_c\rho_l/(\rho_l + \theta\rho_c))$. The production technology is given by $Y_t = Al_t$, again. Because (37) is exactly the same as (16), the equilibrium consumption level and labor supply are given by the two equations of (17) again. In equilibrium, all selves have the same utility level:

$$U_E^\xi = (u(c_E) - v(l_E))/\xi. \tag{38}$$

In equilibrium, the only difference between the non-unitary discount rate model and the model with the time variable discount rate is the difference between $U_E$ and $U_E^\xi$. Let us compare (38) with (19). In the non-unitary discount rate model, the weight on $u(c_E)$, $1/\rho_c$, is different from that on $v(l_E)$, $1/\rho_l$. By contrast, in the model with the time variable discount rate, $u(c_E)$ has the same weight as $v(l_E)$.

We now derive the optimal consumption tax in the model with the time variable discount rate by differentiating (38) with respect to $\tau^c$:

$$\frac{\partial U_E^\xi}{\partial \tau^c} = \frac{1}{c_E \xi}\left(1 - \frac{1}{1+\tau^c}\right)\frac{\partial c_E}{\partial \tau^c} = 0.$$

[16] See Appendix D for the derivations of (36) and (37).

Because $c_E$ is given by the first equation of (17), we have $\partial c_E/\partial\tau^c < 0$ (see (18)). The above equation implies that by setting $\tau^c = 0$, the utility levels of all selves are maximized. This result contrasts with Proposition 4, which shows that in the non-unitary discount rate model, the optimal $\tau^c(= \rho_c/\rho_l - 1)$ is strictly not equal to zero. This exercise reveals that the strictly nonzero optimal consumption tax is not a common feature of general equilibrium models where the problem of time inconsistency arises.

Note that the equilibrium consumption level and labor supply in the two models are exactly the same, and that in equilibrium, the only difference between the two models is whether the weights on $u(c_E)$ and $v(l_E)$ are the same or not. When money is introduced in the same way as in Section 5, we can reasonably conjecture by setting $\rho_c = \rho_l = \xi$ in (33) and (34) that in the model with the time variable discount rate, the zero nominal interest rate (the Friedman rule) becomes optimal and the development of the financial market (a decrease in $\eta$) improves the utility level of all selves.

The analysis in this section provides important policy implications. Even when the problem of time inconsistency exists in the economy, if it is caused by the time variable discount rate, the policy maker might not need to take the problem of time inconsistency into consideration when setting tax rates. However, if the non-unitary discount rates cause the problem of time inconsistency, the policy maker could not implement policy in an appropriate manner if she does not consider the problem of time inconsistency.

# 8 Conclusion

The standard DU model assumes that a single discount rate applies equally to discount (dis)utility from all different sources. However, there is some evidence that people might discount (dis)utility from different sources at different rates. This paper provided a simple model where the agent discounts utility from consumption at a different rate from the disutility of supplying labor.

We first showed that in our non-unitary discount rate model, the preferences of agents are time-inconsistent. The difference between patience concerning consumption and labor is the source of the time inconsistency. Our non-unitary discount rate model may be appropriate for describing the situation where there are (more than) two distinct choice variables that the agent attempts to experience immediately (or procrastinate about), however, she is more willing to experience immediately (or procrastinate about) one of them than the other(s).

In our non-unitary discount rate model, the policy effects on welfare are quite different from the standard models where a single discount rate applies equally to discount (dis)utility from all different sources. For example, when the agent discounts utility from consumption at a higher (lower) rate than the disutility of labor supply, the utility level of agents can improve by a strictly positive consumption tax (a consumption subsidy). We compared our results with those obtained in a time variable discount rate model. Although the preferences are time-inconsistent in both models, the results of welfare analysis are quite different. Our analysis suggested that our results suggests that when the problem of time inconsistency exists in the economy, the optimal policy might be influenced by the sources of time inconsistency.

This paper ignored capital accumulation. The introduction of capital accumulation could affect our results. It is important to examine how our results are affected by the introduction of capital accumulation and to compare our non-unitary discount rate model with a model with a time variable discount rate by considering capital accumulation.

# Appendix

## A. *Preference and Life Satisfaction Survey*

Table 1 is based on micro data from “Preference and Life Satisfaction Survey” conducted in the Global COE Program entitled “Human Behavior and Socioeconomic Dynamics” which is supported by the Ministry of Education, Culture, Sports, Science and Technology in Japan. This survey is a drop-off style survey that was conducted in February and March 2009. The target populations are individuals who are over 20 years old. Sample in the United States was selected randomly from households participating in the managed access panel of TNS (a formerly National Family Opinion), a global market research company. Sample in Japan was selected randomly from all over Japan using the Basic Residents Registration System. Cares were take to ensure that the resulting samples were representative of the total population in both the United State and Japan. Households in samples were mailed questionnaires and were asked to mail them back. The resulting number of respondents were 10708 in the United States and 6181 in Japan.

The question about the money-related discount rate is “Would you choose to receive $100 in two days or to receive a different amount of money in nine days?” If a respondent prefers the receipt of $100 in two days to the receipt of (more than) $100 in nine days, we determine that her money-related discount rate is positive. The question about the labor-related discount rate is “Would you choose to do 60 minutes of labor this Sunday or to do a different minutes of labor next Sunday?” If a respondent prefers doing 60 minutes labor this Sunday to doing (less than) 60 minutes of labor next Sunday, we determine that her labor-related discount rate is negative. If a respondent gave an answer such as he or she prefers 60 minutes of labor this Sunday to 40 minutes of labor next Sunday, which implies that his or her labor-related discount rate is negative, but prefers 80 minutes of labor next Sunday to 60 minutes of labor this Sunday, which implies that his or her labor-related discount rate is positive, we drop him or her from the data because we cannot determine the sign of his or her labor-related discount rate. This also applies to the money-related discount rate. We can determine the signs of both the money- and labor-related discount rates of 6719 (4942) respondents in the United States (Japan). In the United States (Japan), 6202 (4644) respondents, which amounts to about 92% (94%) of 6719 (4942) respondents, were found to discount money-related utility at positive rates. Panels (a) and (b) of Table 1 are based on the data from these 6202 and 4644 respondents, respectively. Table 1 shows that in the United States (Japan), among the 6202 (4644) respondents who discount money-related utility at positive rates, about 70% (74%) of them were found to use negative discount rates to discount the disutility of labor.

## **B.** *Derivation of* (14)

By definition, we have:

$$
\begin{aligned}
\dot{\mu}_t &= \{(1-\tau^r)r_t - g_t^c\}\mu_t - (1+\tau^c),\\
\dot{\nu}_t &= \{(1-\tau^r)r_t - g_t^c\}\nu_t - \theta(1+\tau^c)(\theta\zeta_t)^{\frac{1-\gamma}{\gamma}},\\
\dot{\Phi}_t &= \{\rho_c + (\sigma-1)g_t^c\}\Phi_t - 1,\\
\dot{\Psi}_t &= \{\rho_l + (\sigma-1)g_t^c\}\Phi_t - (\theta\zeta_t)^{\frac{1-\sigma}{\sigma}}.
\end{aligned}
$$

By differentiating both sides of (13) and using the above four equations, we have:

$$\sigma(1+\tau^c)(\Phi_t + \theta\Psi_t)g_t^c = (1-\tau^r)r_t(\mu_t + \nu_t) - (1+\tau^c)(\rho_c\Phi_t + \rho_l\theta\Psi_t).$$

In deriving this equation, we use (13). By using (13), we divide the both sides of the above equation by $\sigma(1+\tau^c)(\Phi_t + \theta\Psi_t)$:

$$g_t^c = \frac{1}{\sigma}\left\{(1-\tau^r)r_t - \frac{\rho_c\Phi_t + \rho_l\theta\Psi_t}{\Phi_t + \theta\Psi_t}\right\}.$$

From this equation and (9), we obtain (14).

## **C.** *General Case: $\sigma \neq \gamma$*

This appendix discusses difficulties that arise when $\gamma$ is not equal to $\sigma$. For simplicity, we assume that $\tau^c$, $\tau^w$ and $\tau^r$ are equal to zero.

The objective of self $t$ and the effects of self $t$'s choices on $a_{t+\Delta}$ are again given by (6) and (7), respectively. As in Section 3, we conjecture that self $v(\geq t+\Delta)$ chooses $c_v$ and $l_v$ so as to satisfy:

$$1 - l_v = (\theta\hat{\zeta}_v)^{\frac{1}{\gamma}} c_v^{\frac{\sigma}{\gamma}}. \tag{39}$$

We conjecture that $\hat{\zeta}_v$ does not depend on the level of asset holdings.

The difficult part of the problem arises from the conjecture as to the effects of $a_{t+\Delta}$ on

$c_v$ where $v \geq t+\Delta$. We assume that each self has incorrect beliefs about the future selves' behavior. More precisely, we assume that self $t$ does not know the effects of her choices on the shape of the path of future consumption. This may be a restrictive assumption. By proceeding with this assumption, however, we can illustrate the difficulties that arise when $\gamma$ is not equal to $\sigma$. Self $t$, however, is assumed to know the effects of her choices on the level of $c_{t+\Delta}$. We conjecture that the path of future consumption is:

$$\hat{g}_v^c \equiv \frac{\dot{c}_v}{c_v} = \frac{1}{\sigma}(r_v - \hat{\omega}_v), \tag{40}$$

As we will see later, $\hat{\omega}_v$ does depend on the level of asset holdings in this general case. Our assumption, however, means that self $t$ does not perceive the effects of the level of asset holdings on $\hat{\omega}_v$.

By using (2), (39) and (40), we obtain the effects of $a_{t+\Delta}$ on $c_{t+\Delta}$:

$$\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}} = \frac{1}{\hat{\mu}_{t+\Delta} + \frac{\sigma}{\gamma}\hat{\nu}_{t+\Delta} c_{t+\Delta}^{\frac{\sigma}{\gamma}-1}}, \tag{41}$$

where $\hat{\mu}_t \equiv \int_t^\infty e^{\int_t^v \{\hat{g}_s^c - r_s\} ds} dv$ and $\hat{\nu}_t \equiv \int_t^\infty w_v(\theta\hat{\zeta}_v)^{\frac{1}{\gamma}} e^{\int_t^v \left\{\frac{\sigma}{\gamma}\hat{g}_s^c - r_s\right\} ds} dv$. Note that self $t$ does not perceive the effects of $a_{t+\Delta}$ on $\hat{\mu}_{t+\Delta}$ and $\hat{\nu}_{t+\Delta}$ because $\hat{\zeta}_v$ does not depend on $a_{t+\Delta}$ and because self $t$ does not perceive the effects of $a_{t+\Delta}$ on $\omega_v$. Therefore, the above equation does not include $\partial\hat{\mu}_{t+\Delta}/\partial a_{t+\Delta}$ and $\partial\hat{\nu}_{t+\Delta}/\partial a_{t+\Delta}$.

By using the policy functions of future selves, (39) and (40), we rewrite the objective function of self $t$, (6), as:

$$U_t = \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{\theta(1-l_t)^{1-\gamma}}{1-\gamma}\right]\Delta + \frac{c_{t+\Delta}^{1-\sigma}}{1-\sigma} e^{-\rho_c\Delta}\hat{\Phi}_{t+\Delta} + \frac{\theta c_{t+\Delta}^{\frac{\sigma(1-\gamma)}{\gamma}}}{1-\gamma} e^{-\rho_l\Delta}\hat{\Psi}_{t+\Delta},$$

where $\hat{\Phi}_t \equiv \int_t^\infty e^{\int_t^v \{(1-\sigma)\hat{g}_u^c - \rho_c\} du} dv$ and $\hat{\Psi}_t \equiv \int_t^\infty (\theta\hat{\zeta}_v)^{\frac{1-\gamma}{\gamma}} e^{\int_t^v \left\{\frac{\sigma(1-\gamma)}{\gamma}\hat{g}_u^c - \rho_l\right\} du} dv$. Self $t$ chooses $c_t$ and $l_t$ so as to maximize this objective function. Note that self $t$ does not perceive the effects of $a_{t+\Delta}$ on $\hat{\Phi}_{t+\Delta}$ and $\hat{\Psi}_{t+\Delta}$ because $\hat{\zeta}_v$ does not depend on $a_{t+\Delta}$ and because self $t$ does not perceive the effects of $a_{t+\Delta}$ on $\omega_v$. As $\Delta$ approaches zero, the first-order conditions become:

$$c_t^{-\sigma} = \hat{X}_t \frac{\partial c_t}{\partial a_t}, \quad \text{and} \quad \theta(1-l_t)^{-\gamma} = w_t \hat{X}_t \frac{\partial c_t}{\partial a_t}, \tag{42}$$

where $\hat{X}_t = c_t^{-\sigma}\hat{\Phi}_t + \frac{\theta\sigma}{\gamma} c_t^{\frac{\sigma(1-\gamma)}{\gamma}-1}\hat{\Psi}_t$.

By following the procedure that we described in Section 3, we obtain:

$$\hat{\zeta}_t = \frac{1}{w_t}, \quad \text{and} \quad \hat{\omega}_t = \frac{\rho_c c_t \hat{\Phi}_t + \rho_l \frac{\theta\sigma}{\gamma} c_t^{\frac{\sigma}{\gamma}} \hat{\Psi}_t}{c_t \hat{\Phi}_t + \frac{\theta\sigma}{\gamma} c_t^{\frac{\sigma}{\gamma}} \hat{\Psi}_t}.$$

When $\gamma$ is equal to $\sigma$, $\hat{\omega}_v$ corresponds to (14). When $\gamma$ is not equal to $\sigma$, $\hat{\omega}_t$ includes $c_t$. Because $c_t$ does depend on $a_t$, $\hat{\omega}_t$ actually depends on the level of asset holdings. In this section, however, we assume that self $t$ does not know the effects of $a_{t+\Delta}$ on $\hat{\omega}_v$. Therefore, under our assumption, there exist intertemporal external effects.

If self $t$ does perceive the dependence of $\hat{\omega}_v$ on $a_{t+\Delta}$ ($v \geq t+\Delta$), we have the following difficulties: The first difficulty arises from the effects of $a_{t+\Delta}$ on $c_{t+\Delta}$. Note that both $\hat{\mu}_{t+\Delta}$ and $\hat{\nu}_{t+\Delta}$ depend on $\hat{\omega}_v$, hence on $a_{t+\Delta}$, through $\hat{g}^c_v = (r_v - \hat{\omega}_v)/\sigma$ ($v \geq t+\Delta$). The effects of $a_{t+\Delta}$ on $c_{t+\Delta}$ through $\hat{\mu}_{t+\Delta}$ and $\hat{\nu}_{t+\Delta}$ are not included in (41). The next difficulty is caused by the dependence of $\hat{\Phi}_{t+\Delta}$ and $\hat{\Psi}_{t+\Delta}$ on $a_{t+\Delta}$. It is apparent that $\hat{\Phi}_{t+\Delta}$ and $\hat{\Psi}_{t+\Delta}$ depend on $a_{t+\Delta}$ because $\hat{\Phi}_{t+\Delta}$ and $\hat{\Psi}_{t+\Delta}$ include $\hat{\omega}_v$ ($v \geq t+\Delta$). If self $t$ does perceive the dependence of $\hat{\omega}_v$ on $a_{t+\Delta}$ ($v \geq t+\Delta$), we have to consider the effects of $a_{t+\Delta}$ on $\hat{\Phi}_{t+\Delta}$ and $\hat{\Psi}_{t+\Delta}$ when maximizing $U_t$. The first-order conditions are no longer given by (42). Because of these difficulties, the problem becomes intractable.

# D. *Derivations of* (36) *and* (37)

Using the same procedure as in Section 3, we derive (36) and (37). Again, the effects of $c_t$ and $l_t$ on $a_{t+\Delta}$ are given by the two equations of (7) if we set $\tau^w = 0$ in (7). As in Section 3, the choices of self $v(\geq t+\Delta)$ and the path of future consumption are conjectured as:

$$1 - l_v = (\theta\chi_v)^{\frac{1}{\sigma}} c_v \quad \text{and} \quad \overline{g}_v^c \equiv \frac{\dot{c}_v}{c_v} = r_v - \xi.$$

As in Section 3, we conjecture that $\chi_v$ and $\xi$ do not depend on the level of asset holdings. Because of the logarithmic utility function, we assume that $\xi$ is constant over time. The effects of $a_{t+\Delta}$ on $c_{t+\Delta}$ are given by:

$$\frac{\partial c_{t+\Delta}}{\partial a_{t+\Delta}} = \frac{1}{\overline{\mu}_{t+\Delta} + \overline{\nu}_{t+\Delta}},$$

where $\overline{\mu}_t \equiv \int_t^\infty (1+\tau^c) e^{\int_t^v \{\overline{g}_s^c - r_s\} ds} dv$ and $\overline{\nu}_t \equiv \int_t^\infty w_v (\theta\chi_v)^{\frac{1}{\sigma}} e^{\int_t^v \{\overline{g}_s^c - r_s\} ds} dv$.

The objective function of self $t$ is given by:

$$U_t = [\ln c_t + \theta \ln(1-l_t)]\,\Delta + (1+\theta)\Omega_{t+\Delta} \ln c_{t+\Delta},$$

where $\Omega_t \equiv \int_t^\infty \exp\{-[\varrho \cdot (v-t) + \phi(v-t)]\}\, dv$. Given the sequence of $\{r_v,\ w_v\}_{v=t}^\infty$, self $t$ chooses $c_t$ and $l_t$ so as to maximize this objective function.

Using the first-order conditions and limiting $\Delta$ to zero, we obtain $\chi_t = (1+\tau^c)/w_t$ and $\xi = 1/\int_0^\infty \exp\{-(\varrho t + \phi(t))\}\, dt$. Then, (36) and (37) are derived.

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(a) United States

| | Labor-related discount rate | |
|---|---|---|
| | negative | positive |
| Money-related discount rate (+) | 4317 | 1885 |
| | (70%) | (30%) |

(b) Japan

| | Labor-related discount rate | |
|---|---|---|
| | negative | positive |
| Money-related discount rate (+) | 3426 | 1218 |
| | (74%) | (26%) |

Table 1. Differences in Discount Rates

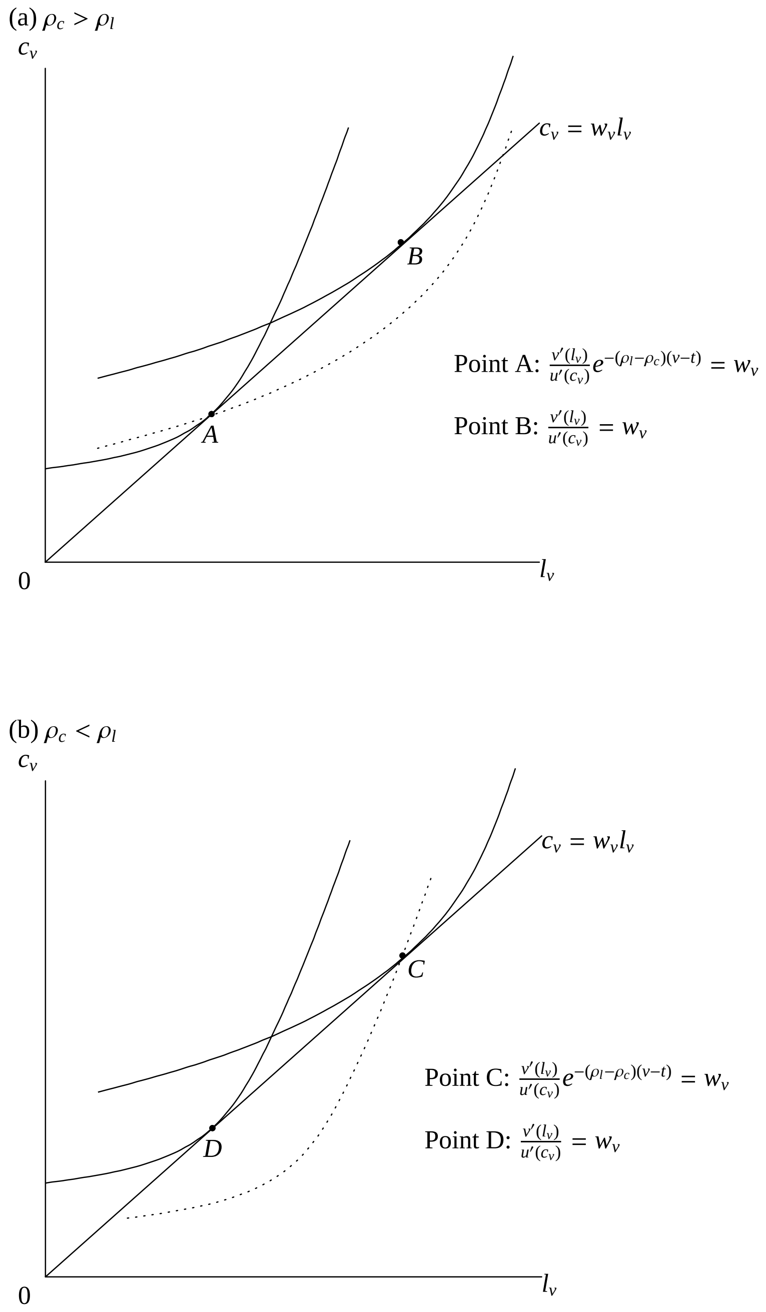


Figure 1. Time Inconsistency of Household's Behavior

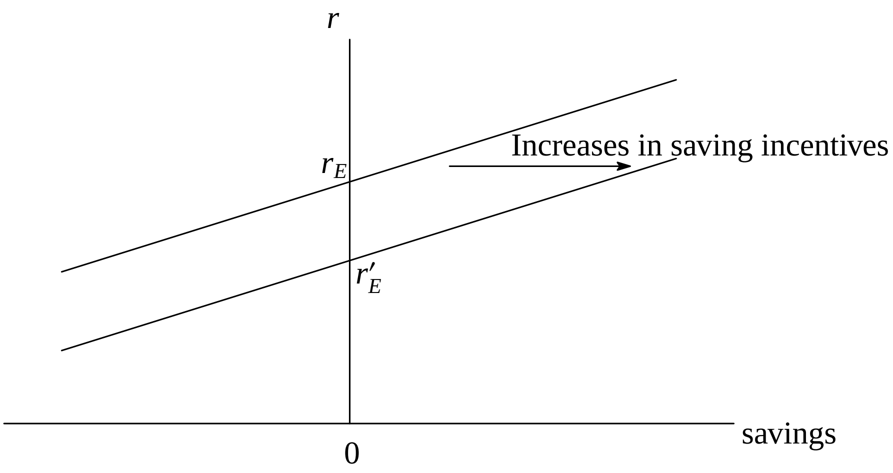


Figure 2. Saving Incentives and Equilibrium Interest Rate