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A Bayesian incentive compatible mechanism for fair division

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Abstract

We consider the problem of fairly allocating one indivisible object when monetary transfers are possible, and examine the existence of *Bayesian incentive compatible* mechanisms to solve the problem. We propose a mechanism that satisfies *envy-freeness*, *budget balancedness*, and *Bayesian incentive compatibility*. Further, we establish the uniqueness of the mechanism under an order additivity condition. This result contrasts well with various results on the incompatibility between efficiency and ex post incentive compatibility.

Keywords: Bayesian incentive compatibility, Fair division, Indivisible good, Mechanism design.

JEL codes: C72, C78, D61, D63, D71.

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1 Introduction

We consider the problem of fairly allocating one indivisible object when monetary transfers are possible. An important example of this problem is that of locating a public facility (e.g., Kleindorfer and Serter, 1994; Sakai, 2008). When a region accepts an undesirable public facility such as a garbage disposal facility, the region must bear a disutility. Here, the problem is which region accepts the facility and how much the site is fairly compensated by the others. Other examples are problems of assigning a right or task.

We examine the existence of *mechanisms* to solve the fair division problem.[1] We discuss this topic in an independent valuation model. In this paper, we mainly consider the following three properties: (ex post) *envy-freeness* (Foley, 1967), (ex post) *budget balancedness*[2], and *Bayesian incentive compatibility*. In our model, under *budget balancedness*, *envy-freeness* implies *efficiency* (Svensson, 1983).[3] This relation encourages us to study the fairness property. In this model, any *efficient* mechanism is not *strategy-proof* (Holmström, 1979; Ohseto, 2000; Schummer, 2000). By the logical relationship, any *envy-free* and *budget balanced* mechanism violates *strategy-proofness*.[4] Therefore, we consider *Bayesian incentive compatibility*, which is a weaker condition than *strategy-proofness*.

We can establish an affirmative result. We propose a mechanism that satisfies *envy-freeness*, *budget balancedness*, and *Bayesian incentive compatibility*. Therefore, we can achieve fairness as well as efficiency through this mechanism. This result contrasts well with the impossibility results on *strategy-proofness*. Further, we formulate *order additivity* as an auxiliary property. Since the monetary transfers in *order additive* mechanisms are not dependent on the names of agents, the property can be interpreted as a kind of anonymity condition. We establish the uniqueness of the desirable mechanism under the order additivity condition. We also discuss an extension of our results to the problem where there are two or more homogeneous indivisible objects.

The paper is organized as follows: Section 2 discusses the related literature. Section 3 introduces our model and the properties of mechanisms. Section 4 presents our main results. Section 5 discusses an extension of our results. Section 6 concludes the paper.

2 Related literature

On the basis of the impossibility results on *strategy-proofness*, many studies consider weaker conditions and examine the compatibility among properties. If we give up *strategy-proofness* or *budget balancedness*, positive results can be obtained: for ex-

[1] A mechanism is a function that associates each preference profile with an allocation. Therefore, it is also considered as a *social choice function* or its *associated direct revelation mechanism*.

[2] Hereafter, we simply call these *envy-freeness* and *budget balancedness*.

[3] In this paper, *efficiency* means ex post Pareto efficiency.

[4] Tadenuma and Thomson (1995) directly prove this relation.

ample, Arrow (1979) and d'Aspremont and Gérard-Varet (1979) for *efficiency* and *Bayesian incentive compatibility*,[5] and Ohseto (2006) for *envy-freeness* and *strategy-proofness*.[6] On the other hand, we cannot overcome the impossibility under a weaker fairness condition than *envy-freeness*, for example *symmetry* and the *identical preferences lower bound* (Moulin, 1990)[7]; Ohseto (1999), Schummer (2000), Ando, Kato, and Ohseto (2008), and Fujinaka and Sakai (2007b).

In this paper, we weaken *strategy-proofness* and establish a positive result. To the best of our knowledge, our paper is the first work to establish the compatibility among the three main properties in our independent valuation model. In this model, Güth and van Damme (1986) and Cramton, Gibbons, and Klemperer (1987) analyze *envy-free* and *budget balanced* mechanisms with simple monetary transfer schemes, but the mechanisms are not *Bayesian incentive compatible*. Morgan (2004) establishes the same compatibility in a common valuation model. However, his result cannot apply to our model since the random assignment of the object in his mechanism violates *efficiency*.[8]

In a complete information model, we can obtain positive results and a key solution is the *no-envy solution*.[9] The solution is the smallest among *Nash implementable* solutions that satisfy *weak symmetry* and an independence property (Tadenuma and Thomson, 1995; Sakai, 2007)[10] and can be implemented in Nash equilibrium through an *envy-free* and *budget balanced* mechanism with an auxiliary condition (Tadenuma and Thomson, 1995; Āzacis, 2008).[11] Fujinaka and Sakai (2008b) propose an *indirect mechanism* that implements the solution in Nash equilibrium. The indirect mechanism also implements the *equal welfare mechanism* (Tadenuma and Thomson, 1993) in *undominated Nash equilibrium*, although any fair mechanism is not Nash implementable (Fujinaka and Sakai, 2007a).

Several studies examine the compatibility among *efficiency*, *interim individual rationality*, and *Bayesian incentive compatibility*; for example, Myerson and Satterthwaite (1983), Cramton, Gibbons, and Klemperer (1987), and Fieseler, Kittsteiner, and Moldovanu (2003). Muto and Oyama (2008) consider *ex post individual rationality* instead of the interim condition, and point out that the mechanism proposed in our paper satisfies the ex post condition for the equal-share ownership setting.[12] They discuss this topic in an interdependent valuation model by applying our mech-

[5] We can easily see that their mechanisms are not *envy-free*.

[6] Any of his mechanisms is not *budget balanced* since it is a Groves mechanism (Groves, 1973).

[7] Moulin (1990) and Beviá (1996) establish that under *budget balancedness*, *envy-freeness* implies the *identical preferences lower bound*.

[8] This is because an agent whose valuation is not the highest may receive the object.

[9] The no-envy solution is a correspondence that associates each preference profile with the set of all *envy-free* and *budget balanced* allocations.

[10] In non-quasi-linear environments, the no-envy solution is the only *Nash implementable* solution that satisfies *symmetry* and an independence property (Sakai, 2007).

[11] Tadenuma and Thomson's (1995) actual purpose is to analyze the degree of strategic manipulability in *envy-free* and *budget balanced* mechanisms. The succeeding studies are Tatamitani (1994), Beviá (2001), and Fujinaka and Sakai (2007b;2008a).

[12] This is because the *identical preferences lower bound* is equivalent to *ex post individual rationality* under *budget balancedness*.

anism to the model.

3 Model

3.1 Basic notion

Let $I \equiv \{1, 2, \ldots, n\}$ be a finite set of the *agents*. There is one indivisible object to be assigned to one of the agents. We assume that monetary transfers are possible.

Agent i has a valuation v_i to the indivisible object. Let $\mathcal{V} \equiv [\underline{v}, \overline{v}]$ with $\underline{v} < \overline{v}$ denote the set of agent i's possible valuations. Each agent's valuations are independently and identically distributed on $\mathcal{V}$ according to the distribution function $F : \mathcal{V} \to [0, 1]$. Suppose that F admits the density function $f \equiv F'$ that satisfies $f(w) > 0$ for each $w \in \mathcal{V}$. Let $v = (v_1, v_2, \ldots, v_n)$ be a valuation profile and $\mathcal{V}^I$ be the set of such profiles. Let $v_{-i} = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ be a valuation profile except for that of agent i and let $\mathcal{V}_{-i}$ be the set of such profiles, i.e., $\mathcal{V}_{-i} \equiv \mathcal{V}^{I \setminus \{i\}}$. We denote the joint density of $v_{-i} \in \mathcal{V}_{-i}$ by $f_{-i}(v_{-i})$. Since the valuations are independently distributed,

$$f_{-i}(v_{-i}) = f(v_1) \times \cdots \times f(v_{i-1}) \times f(v_{i+1}) \times \cdots \times f(v_n) \quad \forall\, v_{-i} \in \mathcal{V}_{-i}.$$

We assume that each agent privately knows his own valuation and only knows that the other agents' valuations are independently distributed according to F.

Agent i with valuation v_i has a utility function $u(\,\cdot\,; v_i) : \{0, 1\} \times \mathbb{R} \to \mathbb{R}$ such that

$$u(x_i, t_i; v_i) \equiv x_i v_i + t_i.$$

$x_i = 1$ (resp. $x_i = 0$) represents that agent i receives (resp. does not receive) the object. $t_i \geq 0$ (resp. $t_i < 0$) is the amount of money he is paid (resp. pays). We assume that each agent is risk neutral.

An *assignment vector* is a vector $x = (x_1, x_2, \ldots, x_n) \in \{0, 1\}^I$ such that $\sum_{i \in I} x_i = 1$. The set of assignment vectors is denoted by X, i.e., $X \equiv \{x \in \{0, 1\}^I : \sum_{i \in I} x_i = 1\}$. A *monetary transfer vector* is a vector $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^I$ such that $\sum_{i \in I} t_i \leq 0$. The set of monetary transfer vectors is denoted by T, i.e., $T \equiv \{t \in \mathbb{R}^I : \sum_{i \in I} t_i \leq 0\}$. An *allocation* (x, t) is a pair of vectors: an assignment vector $x \in X$ and a monetary transfer vector $t \in T$. Let $A \equiv X \times T$ be the set of allocations. Further, let $(x, t) = (x_i, t_i)_{i \in I} \in A$. (x_i, t_i) denotes agent i's consumption bundle.

A *mechanism* $\psi : \mathcal{V}^I \to A$ is a function that associates each valuation profile $v \in \mathcal{V}^I$ with an allocation $\psi(v) = (\psi_i(v))_{i \in I} = (x_i(v), t_i(v))_{i \in I} \in A$. Two mechanisms ψ and ϕ are *equivalent almost everywhere* if the probability measure of the set $\{v \in \mathcal{V}^I : \psi(v) \neq \phi(v)\}$ is equal to zero. ψ and ϕ are *welfare equivalent* if for each $v \in \mathcal{V}^I$ and each $i \in I$, $u(\psi_i(v); v_i) = u(\phi_i(v); v_i)$.

3.2 Properties of mechanisms

In this paper, we are mainly interested with the following three properties: *envy-freeness*, *budget balancedness*, and *Bayesian incentive compatibility.*

We first introduce our fairness requirement. *Envy-freeness* states that every agents weakly prefers his own consumption to that of any other agent (Foley, 1967).

Envy-freeness: A mechanism ψ is *envy-free* if for each $v \in \mathcal{V}^I$ and each $i, j \in I$, $u(\psi_i(v); v_i) \geq u(\psi_j(v); v_i)$.

We also require that no money should be wasted.

Budget balancedness: A mechanism ψ is *budget balanced* if for each $v \in \mathcal{V}^I$, $\sum_{i \in I} t_i(v) = 0$.

The existence of *envy-free* and *budget balanced* mechanisms is guaranteed by the result of Alkan, Demange, and Gale (1991). The following proposition (Tadenuma and Thomson, 1995) identifies the set of such mechanisms in our model. For each $v = (v_1, v_2, \ldots, v_n) \in \mathcal{V}^I$ and each $k = 1, 2, \ldots, n$, let v^k be the kth highest valuation among v.[13]

Proposition 1. A mechanism ψ is *envy-free* and *budget balanced* if and only if for each $v \in \mathcal{V}^I$, letting $x_j(v) = 1$

$$\begin{aligned} v_j &= v^1, \\ \sum_{i \in I} t_i(v) &= 0, \\ \frac{v^2}{n} \leq t_i(v) = t_h(v) &\leq \frac{v^1}{n} \quad \forall i, h \in I \setminus \{j\}. \end{aligned}$$

Proof. We only prove the "only if" part. Let ψ be an *envy-free* and *budget balanced* mechanism and $v \in \mathcal{V}^I$ be a valuation profile. By *envy-freeness*, $t_i(v) = t_h(v)$ for each $i, h \in I \setminus \{j\}$. Let $t_\beta \equiv t_i(v)$ for each $i \in I \setminus \{j\}$. By *budget balancedness*, $t_j(v) = -(n-1)t_\beta$. Since j and each $i \in I \setminus \{j\}$ do not envy each other, $v_j + t_j(v) \geq t_\beta$, and $t_\beta \geq v_i + t_j(v)$. Therefore, we have that $v_j \geq v_i$ and $\frac{v_i}{n} \leq t_\beta \leq \frac{v_j}{n}$. $\square$

Proposition 1 implies that under *budget balancedness*, *envy-freeness* is a refinement of *efficiency* (Svensson, 1983). This relation strengthens our motivation to search for an incentive compatible mechanism that satisfies the fairness property.

We next introduce our incentive compatibility condition. *Bayesian incentive compatibility* states that for each agent, a truthful revelation of his valuation maximizes his expected payoff given that all other agents report their own preferences truthfully.

[13] For example, if $v_1 \geq v_2 \geq \cdots \geq v_n$, then $v^k = v_k$ for each $k = 1, 2, \ldots, n$.

Bayesian incentive compatibility: A mechanism ψ is *Bayesian incentive compatible* if for each $i \in I$ and $v_i \in \mathcal{V}$,

$$\int_{\mathcal{V}_{-i}} u(\psi_i(v_i, v_{-i}); v_i) f_{-i}(v_{-i}) dv_{-i} \geq \int_{\mathcal{V}_{-i}} u(\psi_i(v_i', v_{-i}); v_i) f_{-i}(v_{-i}) dv_{-i},$$

for all $v_i' \in \mathcal{V}$.

In addition to the three properties, we formulate *order additivity* as an auxiliary property. It requires that monetary transfers (i) depend only on $v^k, k = 1, 2, \ldots, n$ and (ii) are additively separable with respect to them. For each $v \in \mathcal{V}^I$ and each $k = 1, 2, \ldots, n$, let $k(v) \in I$ be an agent with the kth highest valuation among v, i.e., $v_{k(v)} = v^k$.

Order additivity: A mechanism ψ is *order additive* if there exist differentiable functions $\tau_\ell^k : \mathcal{V} \to \mathbb{R}$, $k, \ell = 1, 2, \ldots, n$ such that for each $v \in \mathcal{V}^I$ and for each $\ell = 1, 2, \ldots, n$, $t_{\ell(v)}(v) = \tau_\ell^1(v^1) + \tau_\ell^2(v^2) + \cdots + \tau_\ell^n(v^n)$.

Under *order additivity*, monetary transfers are independent of the names of agents. Therefore, it can be interpreted as a kind of anonymity condition. Many of mechanisms in related literature satisfy the property. In the following, we present examples of such mechanisms.

Examples of *order additive* mechanisms : First-price and second-price auction mechanisms obviously satisfy *order additivity* because monetary transfers depend only on v^1 or v^2.

Given $\lambda \in [0, 1]$, let ψ^λ be a mechanism such that for each $v \in \mathcal{V}^I$, letting $x_j^\lambda(v) = 1$,

$$v_j = v^1,$$
$$t_i^\lambda(v) = \begin{cases} -\dfrac{n-1}{n}\left((1-\lambda)v^1 + \lambda v^2\right) & \text{if } i = j \\[2ex] \dfrac{1}{n}\left((1-\lambda)v^1 + \lambda v^2\right) & \text{if } i \in I \setminus \{j\}. \end{cases}$$

It is obvious that ψ^λ satisfies *order additivity.*[14] When $\lambda = 0$, it is an equal welfare mechanism (Tadenuma and Thomson, 1993) in which each agent enjoys the equal utility $\frac{v^1}{n}$.

A mechanism ψ^E is an expected externality mechanism (Arrow, 1979; d'Aspremont and Gérard-Varet, 1979) if for each $v \in \mathcal{V}^I$, letting $x_j^E(v) = 1$,

$$v_j = v^1,$$
$$t_i^E(v) = E_{v_{-i}}\left[W_{-i}(v_i, v_{-i})\right] - \frac{1}{n-1} \sum_{h \neq i} E_{v_{-h}}\left[W_{-h}(v_h, v_{-h})\right] \quad \text{for each } i \in I,$$

[14] Any ψ^λ obviously satisfies *envy-freeness* and *budget balancedness* but does not satisfy *Bayesian incentive compatibility* (Güth and van Damme, 1986; Cramton, Gibbons, and Klemperer, 1987).

where $W_{-i}(v) \equiv \sum_{h \neq i} v_h x_h^E(v)$ for each $i \in I$ and $E_{v_{-i}}[\cdot]$ denotes the expectation operator with respect to v_{-i}.[15] We can verify that it is *order additive* by defining for each $\ell = 1, 2, \ldots, n$,

$$\tau_\ell^\ell(v^\ell) = E_{v_{-\ell(v)}}\left[W_{-\ell(v)}(v^\ell, v_{-\ell(v)})\right]$$
$$\tau_\ell^k(v^k) = -\frac{1}{n-1} E_{v_{-k(v)}}\left[W_{-k(v)}(v^k, v_{-k(v)})\right] \quad \text{for each } k \neq \ell.$$

Other examples are follows: a mechanism in McAfee and McMillan (1992) that is implemented through the first-price knockout preauction[16] and a mechanism associated with a Shapley allocation (Shapley, 1953; Littlechild and Owen, 1973).

4 Double cumulation mechanism

Our purpose is to design a mechanism that satisfies *envy-freeness*, *budget balancedness*, and *Bayesian incentive compatibility.* For simplicity, we restrict our attention to the class of *order additive* mechanisms.

For our purpose, we first examine the implications of the attractive properties. The following lemma states that if an *order additive* mechanism satisfies *envy-freeness* and *budget balancedness*, it has a considerably simpler monetary transfer scheme. The proof of the lemma can be found in the Appendix.

Lemma 1. If an *order additive* mechanism ψ satisfies *envy-freeness* and *budget balancedness*, then there are two functions $\tau^1 : \mathcal{V} \to \mathbb{R}$ and $\tau^2 : \mathcal{V} \to \mathbb{R}$ that satisfy the following: for each $v \in \mathcal{V}^I$ and each $i \in I$ such that $x_i(v) = 0$,

$$t_i(v) = \tau^1(v^1) + \tau^2(v^2).$$

We next establish that under *order additivity,* the following mechanisms are the only candidates for desirable mechanisms. We term these *double cumulation mechanisms* since the monetary transfers depend on the cumulation of the cumulative distribution function of each agent's valuation, $\int_{\underline{v}}^{v_i} F(w)dw$. Note that later, we will also establish that any double cumulation mechanism actually satisfies the three desirable properties.

[15] ψ^E satisfies *efficiency* and *Bayesian incentive compatibility* (Arrow, 1979;d'Aspremont and Gérard-Varet, 1979) but does not satisfy *envy-freeness.*

[16] See McAfee and McMillan (1992) for details regarding the first-price knockout pre-auction.

Double cumulation mechanism: A mechanism ψ is a double cumulation mechanism if for each $v \in \mathcal{V}^I$, letting $x_j(v) = 1$,

$$v_j = v^1,$$
$$t_i(v) = \begin{cases} -\dfrac{n-1}{n}\left(v^1 - \displaystyle\int_{v^2}^{v^1} F(w)dw\right) & \text{if } i = j \\ \dfrac{1}{n}\left(v^1 - \displaystyle\int_{v^2}^{v^1} F(w)dw\right) & \text{if } i \in I \setminus \{j\}. \end{cases}$$

The double cumulation mechanism is not unique since the assignment of the object cannot be uniquely determined by the definition. However, any two double cumulation mechanisms are equivalent almost everywhere. Further, the mechanisms are welfare equivalent. Therefore, the double cumulation mechanism is essentially unique.

Theorem 1. If a mechanism satisfies *envy-freeness*, *budget balancedness*, *Bayesian incentive compatibility*, and *order additivity*, then it is a double cumulation mechanism.

Proof. We only prove that for each $v \in \mathcal{V}^I$ and each $i \in I$ such that $x_i(v) = 0$,

$$t_i(v) = \frac{1}{n}\left(v^1 - \int_{v^2}^{v^1} F(w)dw\right).$$

By Proposition 1 and Lemma 1, for each $v \in \mathcal{V}^I$ and each $i \in I$ such that $x_i(v) = 0$,

$$\frac{v^2}{n} \leq t_i(v) = \tau^1(v^1) + \tau^2(v^2) \leq \frac{v^1}{n}. \tag{1}$$

Since the budget is balanced, the accepter pays $(n-1)\left(\tau^1(v^1) + \tau^2(v^2)\right)$ in total. (1) implies that for $w \in \mathcal{V}$, if $v^1 = v^2 = w$,

$$\tau^1(w) + \tau^2(w) = \frac{w}{n}. \tag{2}$$

Furthermore, by differentiating (2), we obtain

$$\frac{d\tau^1(w)}{dv^1} + \frac{d\tau^2(w)}{dv^2} = \frac{1}{n} \quad \forall\, w \in \mathcal{V}. \tag{3}$$

We introduce some definitions. Given $i \in I$, let y_1 or y_2 be the highest or the second highest order statistic of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$ respectively. We denote G or g as the distribution function or the density function of y_1 respectively, i.e.,

$G(w) \equiv F(w)^{n-1}$ and $g(w) \equiv G'(w) = (n-1)f(w)F(w)^{n-2}$ for each $w \in \mathcal{V}$. Let $f(y_1, y_2)$ denote the joint density function of y_1 and y_2, i.e.,

$$f(y_1, y_2) = (n-1)(n-2)f(y_1)f(y_2)F(y_2)^{n-3}$$

if $y_1 \geq y_2$, and 0 otherwise.[17]

Let $\hat{v}_i \in \mathcal{V}$ be the true valuation of agent i. We denote $U(v_i; \hat{v}_i)$ as his expected utility when he reports v_i. The expected utility can be written as follows:

$$\begin{aligned} U(v_i; \hat{v}_i) &= \int_{\mathcal{V}_{-i}} u(\psi_i(v_i, v_{-i}); \hat{v}_i) f_{-i}(v_{-i}) dv_{-i} \\ &= \int_{\underline{v}}^{v_i} \int_{\underline{v}}^{y_1} [\hat{v}_i - (n-1)(\tau^1(v_i) + \tau^2(y_1))] f(y_1, y_2) dy_2 dy_1 \\ &\quad + \int_{v_i}^{\overline{v}} \int_{\underline{v}}^{v_i} (\tau^1(y_1) + \tau^2(v_i)) f(y_1, y_2) dy_2 dy_1 \\ &\quad + \int_{v_i}^{\overline{v}} \int_{v_i}^{y_1} (\tau^1(y_1) + \tau^2(y_2)) f(y_1, y_2) dy_2 dy_1. \end{aligned} \tag{4}$$

By differentiating (4), we have that

$$\begin{aligned} \frac{d}{dv_i} U(v_i; \hat{v}_i) = &-(n-1)\frac{d\tau^1(v_i)}{dv^1} G(v_i) + \frac{d\tau^2(v_i)}{dv^2}(n-1)(1-F(v_i))F(v_i)^{n-2} \\ &+ \left(\hat{v}_i - (n-1)\frac{v_i}{n}\right) g(v_i) - \frac{v_i}{n} g(v_i). \end{aligned} \tag{5}$$

The derivation of (5) can be found in the Appendix. (5) and *Bayesian incentive compatibility* together imply that

$$\begin{aligned} \frac{d}{dv_i} U(\hat{v}_i; \hat{v}_i) &= -(n-1)\frac{d\tau^1(\hat{v}_i)}{dv^1} G(\hat{v}_i) + \frac{d\tau^2(\hat{v}_i)}{dv^2}(n-1)(1-F(\hat{v}_i))F(\hat{v}_i)^{n-2} \\ &\quad + \left(\hat{v}_i - (n-1)\frac{\hat{v}_i}{n}\right) g(\hat{v}_i) - \frac{\hat{v}_i}{n} g(\hat{v}_i) \\ &= -(n-1)\frac{d\tau^1(\hat{v}_i)}{dv^1} G(\hat{v}_i) + \frac{d\tau^2(\hat{v}_i)}{dv^2}(n-1)(1-F(\hat{v}_i))F(\hat{v}_i)^{n-2} = 0. \end{aligned} \tag{6}$$

It is interesting that the F.O.C. (6) depends only on the two terms that represent the expected changes of monetary transfers due to a slight increase of the agent's report. This fact will be thoroughly discussed in Remark 1.

From (3) and (6), it follows that for each $\hat{v}_i \in \mathcal{V}$,

$$\frac{d\tau^1(\hat{v}_i)}{dv^1} = \frac{1}{n}(1 - F(\hat{v}_i)) \text{ and } \frac{d\tau^2(\hat{v}_i)}{dv^2} = \frac{1}{n}F(\hat{v}_i).$$

[17] See Hogg and Craig (1995, pp.193–203) for the definition of the joint density function of y_1 and y_2.

Further, we obtain

$$\tau^1(\hat{v}_i) = \tau^1(\underline{v}) + \frac{1}{n}\int_{\underline{v}}^{\hat{v}_i} (1 - F(w))\, dw = \tau^1(\underline{v}) - \frac{\underline{v}}{n} + \frac{1}{n}\left(\hat{v}_i - \int_{\underline{v}}^{\hat{v}_i} F(w)dw\right),$$

$$\tau^2(\hat{v}_i) = \tau^2(\underline{v}) + \frac{1}{n}\int_{\underline{v}}^{\hat{v}_i} F(w)dw.$$

Therefore, for each $v \in \mathcal{V}^I$ and each $i \in I$ such that $x_i(v) = 0$,

$$\begin{aligned} t_i(v) &= \tau^1(\underline{v}) - \frac{\underline{v}}{n} + \frac{1}{n}\left(v^1 - \int_{\underline{v}}^{v^1} F(w)dw\right) + \tau^2(\underline{v}) + \frac{1}{n}\int_{\underline{v}}^{v^2} F(w)dw \\ &= \frac{1}{n}\left(v^1 - \int_{v^2}^{v^1} F(w)dw\right). \end{aligned}$$

where the second equality is obtained by (2). □

Remark 1. In this remark, we would like to discuss why the F.O.C. (6) depends only on the two terms. We first examine the derivative of $U(v_i; \hat{v}_i)$, (5):

$$\begin{aligned} \frac{d}{dv_i}U(v_i; \hat{v}_i) = &-(n-1)\frac{d\tau^1(v_i)}{dv^1}G(v_i) + \frac{d\tau^2(v_i)}{dv^2}(n-1)(1 - F(v_i))F(v_i)^{n-2} \\ &+ \left(\hat{v}_i - (n-1)\frac{v_i}{n}\right) g(v_i) - \frac{v_i}{n}g(v_i). \end{aligned} \tag{5}$$

A slight increase of the agent's report v_i causes two types of changes: (i) a change in monetary transfers and (ii) a change in the order of v_i. The first or second term of (5) represents the expected change in monetary transfers when his report is the highest or the second highest respectively. The third and fourth terms of (5) represent the expected gains or losses due to a change in the order of his report. When y_1 is just equal to v_i, his report can become the highest valuation by the slight increase of it. Therefore, he slightly increases the chance that his report is the highest. He then receives the object and pays $\frac{v_i}{n}$ to each non-accepter by *envy-freeness*. The third term represents the expected utility due to the slight increase of the chance that v_i is the highest. On the other hand, at the same event, his report v_i cannot be the second highest valuation by the slight increase. Therefore, he slightly decreases the chance that his report is the second highest. He then only receives $\frac{v_i}{n}$ units of money from the accepter by *envy-freeness*. The fourth term represents the expected utility due to the slight decrease of the chance that v_i is the second highest. By (5) and *Bayesian incentive compatibility*,

$$\begin{aligned} \frac{d}{dv_i}U(\hat{v}_i; \hat{v}_i) &= -(n-1)\frac{d\tau^1(\hat{v}_i)}{dv^1}G(\hat{v}_i) + \frac{d\tau^2(\hat{v}_i)}{dv^2}(n-1)(1 - F(\hat{v}_i))F(\hat{v}_i)^{n-2} \\ &\quad + \left(\hat{v}_i - (n-1)\frac{\hat{v}_i}{n}\right) g(\hat{v}_i) - \frac{\hat{v}_i}{n}g(\hat{v}_i) \\ &= -(n-1)\frac{d\tau^1(\hat{v}_i)}{dv^1}G(\hat{v}_i) + \frac{d\tau^2(\hat{v}_i)}{dv^2}(n-1)(1 - F(\hat{v}_i))F(\hat{v}_i)^{n-2} = 0. \end{aligned} \tag{6}$$

The two effects by the changes of $\hat{v}_i$'s order are offset because he enjoys $\frac{\hat{v}_i}{n}$ utility at each event by *envy-freeness*. Therefore, the F.O.C. depends only on the expected changes of monetary transfers.

Any double cumulation mechanism obviously satisfies *order additivity*, and in fact, the three desirable properties as well.

Theorem 2. Any double cumulation mechanism satisfies *envy-freeness*, *budget balancedness*, and *Bayesian incentive compatibility.*

Proof. Let ψ be a double cumulation mechanism. ψ obviously satisfies *budget balancedness.*
(*Envy-freeness*) Since $v^1 \geq v^2$, it holds that $0 \leq \int_{v^2}^{v^1} F(w)dw \leq v^1 - v^2$. This implies that for each $i \in I$ such that $x_i(v) = 0$,

$$\frac{v^2}{n} = \frac{v^1}{n} - \frac{1}{n}(v^1 - v^2) \leq \frac{v^1}{n} - \frac{1}{n}\int_{v^2}^{v^1} F(w)dw = t_i(v) \leq \frac{v^1}{n}.$$

Therefore, by Proposition 1, ψ satisfies *envy-freeness.*
(*Bayesian incentive compatibility*) For each $z \in \mathcal{V}$, let

$$\theta^1(z) \equiv \frac{1}{n}\left(z - \int_{\underline{v}}^{z} F(w)dw\right) \text{ and } \theta^2(z) \equiv \frac{1}{n}\int_{\underline{v}}^{z} F(w)dw.$$

ψ satisfies *envy-freeness*, *budget balancedness*, and *order additivity*; hence, from an argument similar to that in Theorem 1, it follows that

$$\begin{aligned}\frac{d}{dv_i}U(v_i;\hat{v}_i) &= -(n-1)\frac{d\theta^1(v_i)}{dv^1}G(v_i) + \frac{d\theta^2(v_i)}{dv^2}(n-1)(1-F(v_i))F(v_i)^{n-2} \\ &\quad + \left(\hat{v}_i - (n-1)\frac{v_i}{n}\right)g(v_i) - \frac{v_i}{n}g(v_i) \\ &= (\hat{v}_i - v_i)g(v_i).\end{aligned}$$

where the second equality is obtained by $\frac{d\theta^1(v_i)}{dv^1} = \frac{1}{n}\left(1 - F(v_i)\right)$ and $\frac{d\theta^2(v_i)}{dv^2} = \frac{1}{n}F(v_i)$. Therefore, we have that

$$\begin{aligned}&\frac{d}{dv_i}U(\hat{v}_i;\hat{v}_i) = (\hat{v}_i - \hat{v}_i)g(\hat{v}_i) = 0, \\ &\frac{d^2}{dv_i^2}U(\hat{v}_i;\hat{v}_i) = (\hat{v}_i - \hat{v}_i)g'(\hat{v}_i) - g'(\hat{v}_i) = -g'(\hat{v}_i) < 0.\end{aligned}$$

Thus, ψ satisfies *Bayesian incentive compatibility.* □

We can immediately establish the following characterization theorem on the basis of Theorems 1 and 2.

Theorem 3. A mechanism satisfies *envy-freeness*, *budget balancedness*, *Bayesian incentive compatibility*, and *order additivity* if and only if it is a double cumulation mechanism.

As we stated above, the double cumulation mechanism is essentially unique. Therefore, Theorem 3 indicates that under *order additivity*, a mechanism that satisfies the desirable properties is essentially unique.

5 Extension

In the preceding sections, we consider the problem with one indivisible object. In this section, we discuss the problem where there are several homogenous indivisible objects. Our results in the previous section can be applied to the problem straightforwardly.[18]

We assume that there are m ($1 \leq m < n$) units of homogeneous indivisible object to be allocated and each agent consumes at most one object. In this setting, an *assignment vector* is a vector $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^I$ such that $\sum_{i \in I} x_i = m$.

The following proposition is obtained from Lemma 2 in Bochet and Sakai (2007).[19]

Proposition 2. A mechanism ψ is *envy-free* and *budget balanced* if and only if for each $v \in \mathcal{V}^I$, letting $I^\alpha(v) \equiv \{i \in I : x_i(v) = 1\}$,

$$\begin{aligned}
& v_i \geq v^m \geq v_j \quad \forall\, i \in I^\alpha(v), \forall\, j \in I \setminus I^\alpha(v), \\
& \sum_{i \in I} t_i(v) = 0, \\
& t_i(v) = t_j(v) \quad \forall\, i, j \in I^\alpha(v), \\
& \frac{m}{n} v^{m+1} \leq t_i(v) = t_j(v) \leq \frac{m}{n} v^m \quad \forall\, i, j \in I \setminus I^\alpha(v).
\end{aligned}$$

The following lemma follows from an argument similar to Lemma 1.

Lemma 2. If an *order additive* mechanism ψ satisfies *envy-freeness* and *budget balancedness*, then there are two functions $\tau^m : \mathcal{V} \to \mathbb{R}$ and $\tau^{m+1} : \mathcal{V} \to \mathbb{R}$ that satisfy the following: for each $v \in \mathcal{V}^I$, letting $I^\alpha(v) \equiv \{i \in I : x_i(v) = 1\}$,

$$t_i(v) = \tau^m(v^m) + \tau^{m+1}(v^{m+1}) \quad \forall\, i \in I \setminus I^\alpha(v).$$

We can also establish that the double cumulation mechanisms are the only ones that satisfy the four properties: a mechanism ψ is a double cumulation mechanism if for each $v \in \mathcal{V}^I$, letting $I^\alpha(v) \equiv \{i \in I : x_i(v) = 1\}$,

$$\begin{aligned}
& v_i \geq v^m \geq v_j \quad \forall\, i \in I^\alpha(v), \ \forall\, j \in I \setminus I^\alpha(v) \\
& t_i(v) = \begin{cases} -\dfrac{n-m}{n}\left(v^m - \displaystyle\int_{v^{m+1}}^{v^m} F(w)dw\right) & \text{if } \ i \in I^\alpha(v) \\[2ex] \dfrac{m}{n}\left(v^m - \displaystyle\int_{v^{m+1}}^{v^m} F(w)dw\right) & \text{if } \ i \in I \setminus I^\alpha(v). \end{cases}
\end{aligned}$$

Theorem 4. In the case where there are several homogenous indivisible objects, a mechanism satisfies *envy-freeness*, *budget balancedness*, *Bayesian incentive compatibility*, and *order additivity* if and only if it is a double cumulation mechanism.

[18] The author would like to thank Minoru Kitahara for pointing out this fact. Discussions in this section are based on his comments.

[19] See Bochet and Sakai (2007) for the details of the proof.

Proof. We only prove the "only if" part. Let ψ be a mechanism that satisfies the four properties. By Proposition 2 and Lemma 2, for each $v \in \mathcal{V}^I$ and each $w \in \mathcal{V}$, if $v^m = v^{m+1} = w$,

$$\tau^m(w) + \tau^{m+1}(w) = \frac{m}{n}w.$$

By differentiating this, we have that

$$\frac{d\tau^m(w)}{dv^m} + \frac{d\tau^{m+1}(w)}{dv^{m+1}} = \frac{m}{n}. \tag{7}$$

Similar to the discussion in Remark 1, we can see that the F.O.C. depends only on the expected change of monetary transfers. Therefore,

$$\begin{aligned} \frac{d}{dv_i}U(\hat{v}_i;\hat{v}_i) = &-\frac{n-m}{m}\frac{d\tau^m(\hat{v}_i)}{dv^m}\frac{(n-1)!}{(m-1)!(n-m)!}(1-F(\hat{v}_i))^{m-1}F(\hat{v}_i)^{n-m} \\ &+\frac{d\tau^{m+1}(\hat{v}_i)}{dv^{m+1}}\frac{(n-1)!}{m!(n-m-1)!}(1-F(\hat{v}_i))^{m}F(\hat{v}_i)^{n-m-1} = 0 \end{aligned} \tag{8}$$

where $\frac{(n-1)!}{(m-1)!(n-m)!}(1-F(\hat{v}_i))^{m-1}F(\hat{v}_i)^{n-m}$ or $\frac{(n-1)!}{m!(n-m-1)!}(1-F(\hat{v}_i))^{m}F(\hat{v}_i)^{n-m-1}$ is the probability that $\hat{v}_i$ is the mth or $(m+1)$th highest valuation, respectively. By (7) and (8), we can obtain that for each $v \in \mathcal{V}^I$ and each $i \in I$ such that $x_i(v) = 0$,

$$t_i(v) = \tau^m(v^m) + \tau^{m+1}(v^{m+1}) = \frac{m}{n}\left(v^m - \int_{v^{m+1}}^{v^m} F(w)dw\right).$$

Therefore, ψ is a double cumulation mechanism. $\square$

6 Concluding remarks

In this paper, we consider the problem of fairly allocating one indivisible object when monetary transfers are possible. We established the existence of a *Bayesian incentive compatible* mechanism to solve the problem and the uniqueness of the mechanism under an additive separability condition. This is a positive result in contrast to various impossibility results on *strategy-proofness* although we consider a weaker incentive compatibility condition.

In this paper, we restrict our analysis to the class of *order additive* mechanisms. We have not found a *non order additive* mechanism that satisfies the three main properties. It is a very interesting question whether there is a desirable mechanism that is not *order additive.*

The double cumulation mechanism relies on the distribution function of each agent's valuation, F. Therefore, it is not a *simple* mechanism, which is one that does not depend on a specific information of the environment (Cramton, Gibbons, and Klemperer, 1987; McAfee, 1992). It is interesting to address an issue of designing a simple direct or indirect mechanism to achieve fair division in a model with incomplete information.

Appendix

Proof of Lemma 1

For each $v \in \mathcal{V}^I$, let $t_\beta(v)$ be the amount of money that each non-accepter receives. By *order additivity*, there exist functions $\tau^k : \mathcal{V} \to \mathbb{R}$, $k = 1, 2, \dots, n$ such that for each $v \in \mathcal{V}^I$, $t_\beta(v) = \tau^1(v^1) + \tau^2(v^2) + \cdots + \tau^n(v^n)$. From Proposition 1, it follows that for each $v \in \mathcal{V}^I$,

$$\frac{v^2}{n} \leq t_\beta(v) = \tau^1(v^1) + \tau^2(v^2) + \cdots + \tau^n(v^n) \leq \frac{v^1}{n}. \tag{9}$$

For each $w, w' \in \mathcal{V}$, let us consider the valuation profiles $v = (\overline{v}, \overline{v}, \dots, \overline{v}, w)$ and $v' = (\overline{v}, \overline{v}, \dots, \overline{v}, w')$. By (9),

$$\begin{aligned} t_\beta(v) &= \tau^1(\overline{v}) + \tau^2(\overline{v}) + \cdots + \tau^{n-1}(\overline{v}) + \tau^n(w) = \frac{\overline{v}}{n} \\ t_\beta(v') &= \tau^1(\overline{v}) + \tau^2(\overline{v}) + \cdots + \tau^{n-1}(\overline{v}) + \tau^n(w') = \frac{\overline{v}}{n}. \end{aligned}$$

These imply that $\tau^n(w) = \tau^n(w')$ for each $w, w' \in \mathcal{V}$. Therefore, τ^n is constant, i.e., $\tau^n(w) = c_n \in \mathbb{R}$ for each $w \in \mathcal{V}$. By repeating similar arguments, we can obtain that for each $k = 3, 4, \dots, n-1$, τ^k is constant, i.e., $\tau^k(w) = c_k \in \mathbb{R}$ for each $w \in \mathcal{V}$. Thus, for each $v \in \mathcal{V}^I$,

$$t_\beta(v) = \tau^1(v^1) + \tau^2(v^2) + \sum_{k=3}^{n} c_k.$$

□

Derivation of (5)

Let

$$\begin{aligned} U^1(v_i; \hat{v}_i) &\equiv \int_{\underline{v}}^{v_i} \int_{\underline{v}}^{y_1} [\hat{v}_i - (n-1)(\tau^1(v_i) + \tau^2(y_1))] f(y_1, y_2) dy_2 dy_1 \\ U^2(v_i; \hat{v}_i) &\equiv \int_{v_i}^{\overline{v}} \int_{\underline{v}}^{v_i} (\tau^1(y_1) + \tau^2(v_i)) f(y_1, y_2) dy_2 dy_1 \\ U^3(v_i; \hat{v}_i) &\equiv \int_{v_i}^{\overline{v}} \int_{v_i}^{y_1} (\tau^1(y_1) + \tau^2(y_2)) f(y_1, y_2) dy_2 dy_1. \end{aligned}$$

Recall that by (2), for each $w \in \mathcal{V}$, $\tau^1(w)+\tau^2(w) = \frac{w}{n}$. We can obtain the followings:

$$
\begin{aligned}
\frac{d}{dv_i}U^1(v_i;\hat{v}_i) = &\int_{\underline{v}}^{v_i} \frac{\partial}{\partial v_i}\left[\int_{\underline{v}}^{y_1} \left[\hat{v}_i - (n-1)\big(\tau^1(v_i)+\tau^2(y_1)\big)\right] f(y_1,y_2)dy_2\right] dy_1 \\
&+ \int_{\underline{v}}^{v_i} \left[\hat{v}_i - (n-1)\big(\tau^1(v_i)+\tau^2(v_i)\big)\right] f(v_i,y_2)dy_2 \\
= &\int_{\underline{v}}^{v_i}\int_{\underline{v}}^{y_1} \frac{\partial}{\partial v_i}\Big[\left[\hat{v}_i - (n-1)\big(\tau^1(v_i)+\tau^2(y_1)\big)\right] f(y_1,y_2)\Big] dy_2 dy_1 \\
&+ \int_{\underline{v}}^{v_i} \left[\hat{v}_i - (n-1)\frac{v_i}{n}\right] f(v_i,y_2)dy_2 \\
= &-(n-1)\frac{d\tau^1(v_i)}{dv^1}\int_{\underline{v}}^{v_i}\int_{\underline{v}}^{y_1} f(y_1,y_2)dy_2dy_1 + \left(\hat{v}_i - (n-1)\frac{v_i}{n}\right) g(v_i) \\
= &-(n-1)\frac{d\tau^1(v_i)}{dv^1}G(v_i) + \left(\hat{v}_i - (n-1)\frac{v_i}{n}\right) g(v_i). \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dv_i}U^2(v_i;\hat{v}_i) \\
&= \int_{v_i}^{\overline{v}} \frac{\partial}{\partial v_i}\left[\int_{\underline{v}}^{v_i} (\tau^1(y_1)+\tau^2(v_i))f(y_1,y_2)dy_2\right] dy_1 - \int_{\underline{v}}^{v_i}(\tau^1(v_i)+\tau^2(v_i))f(v_i,y_2)dy_2 \\
&= \int_{v_i}^{\overline{v}} \left[\int_{\underline{v}}^{v_i} \frac{\partial}{\partial v_i}\Big((\tau^1(y_1)+\tau^2(v_i))f(y_1,y_2)\Big)dy_2 + (\tau^1(y_1)+\tau^2(v_i))f(y_1,v_i)\right] dy_1 - \frac{v_i}{n}g(v_i) \\
&= \frac{d\tau^2(v_i)}{dv^2}\int_{v_i}^{\overline{v}}\int_{\underline{v}}^{v_i} f(y_1,y_2)dy_2dy_1 + \int_{v_i}^{\overline{v}}(\tau^1(y_1)+\tau^2(v_i))f(y_1,v_i)dy_1 - \frac{v_i}{n}g(v_i) \\
&= \frac{d\tau^2(v_i)}{dv^2}(n-1)(1-F(v_i))F(v_i)^{n-2} + \int_{v_i}^{\overline{v}}(\tau^1(y_1)+\tau^2(v_i))f(y_1,v_i)dy_1 - \frac{v_i}{n}g(v_i) \\
&\qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d}{dv_i}U^3(v_i;\hat{v}_i) \\
&= \int_{v_i}^{\overline{v}} \frac{\partial}{\partial v_i}\left[\int_{v_i}^{y_1} (\tau^1(y_1)+\tau^2(y_2))f(y_1,y_2)dy_2\right] dy_1 - \int_{v_i}^{v_i}(\tau^1(v_i)+\tau^2(y_2))f(v_i,y_2)dy_2 \\
&= \int_{v_i}^{\overline{v}} \left[\int_{v_i}^{y_1} \frac{\partial}{\partial v_i}\Big((\tau^1(y_1)+\tau^2(y_2))f(y_1,y_2)\Big)dy_2 - (\tau^1(y_1)+\tau^2(v_i))f(y_1,v_i)\right] dy_1 \\
&= -\int_{v_i}^{\overline{v}}(\tau^1(y_1)+\tau^2(v_i))f(y_1,v_i)dy_1 \qquad (12)
\end{aligned}
$$

Therefore, (5) immediately follows from (10), (11), and (12). ■

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