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All-Pay Auctions with Handicaps

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Abstract

This paper analyzes an all-pay auction where the winner is determined according to the sum of the bid and a handicap endowed to all players. The bidding strategy in equilibrium is then explicitly derived as a "piecewise affine transformation" of the equilibrium strategy in an all-pay auction without handicaps. The paper also discusses the allocation rule implemented in the equilibrium and provides a comparison of revenue.

Journal of Economic Literature Classification Numbers: D44, D82, D86.

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1 Introduction

In many economic environments, agents compete by expending resources to win prizes. Typical examples include lobbying, bribery, promotion contests, R&D races, economists competing based on the number of publications, and so on. In the auction theory literature, such situations are considered using the model of an all-pay auction where different players spend money or exert effort as "bids" and the player that spends the most "wins the auction" and obtains the final prize. As the "bid" in these models is a sunk cost borne by all players, regardless of the ultimate winner, they are known as "all-pay" auctions.[1][2]

In reality, it is often the case that some players have an advantage over others at the beginning. For instance, the success of lobbying today depends not only on the activity undertaken today but also on the accumulation of past efforts. Likewise, hereditary advantages also play some role in bribery, while the sponsor of a contest probably undertakes some discriminatory treatment in favor of one group of the players. From the viewpoint of auction theory, such (dis)advantages are regarded as "handicaps" in the all-pay auction; that is, the highest bidder is not always the winner, as the player whose bid amount less his "handicap" is the highest wins the auction. The present paper is devoted to the study of an all-pay auction where the advantages of players are described as handicaps existing among agents.[3]

[1]Amann and Leininger [1] and Krishna and Morgan [11] consider all-pay auctions with incomplete information while Noussair and Silver [21] conduct experimental work. The recent theoretical work on all-pay auctions includes Parreiras and Rubinchik [22].

[2]Ando [2], Che and Gale [4, 5], Kaplan and Wettstein [9], and Moldovanu and Sela [19, 20] apply the all-pay auction model to contests, with recent theoretical work including Siegel [25].

[3]In recent work, Mares and Swinkels [13, 14] study the first price handicap auction

The paper presents two main results in Section 3. The first of these is examination of the entry–exit decisions of players. When there are handicaps in the auction, there is a positive probability that a disadvantaged player finds it optimal to bid zero when his value is very small.[4] Upon consideration, an advantaged player also finds it beneficial to lower his bid; that is, the advantaged player has some positive probability of winning even when he bids zero. Two examples in Section 3.1 illustrate these strategies in the equilibrium. In the examples, each player has an "entry point", and the player bids zero if his value is lower than this point, whereas he makes some positive bid otherwise. A more interesting problem is if there is some case where a player with a heavy handicap totally abandons winning (i.e., he always bids zero regardless of the value). Example 2 in Section 3.1 depicts the case where a player with a heavy handicap adopts a strategy of bidding zero for all possible values (in other words, his entry point is set to the upper bound of the value).

The second result found is a simple derivation of the equilibrium strategy. Theoretically, it is not obvious if the analysis of asymmetric auctions is straightforward. Indeed, it is generally understood in the literature that asymmetries in some auctions lead to numerous complications. For instance, although an equilibrium exists in asymmetric first-price auctions,[5]

(FPHA) along with other asymmetric settings. Our concept of a handicap is the same as in their paper; namely, we treat bids and handicaps in terms of summation. We may also consider another form of "handicap" where the bid of the disadvantaged player is discounted through multiplication (for the two-player case of multiplicative handicaps, see Feess et al. [7]). However, in the present paper we restrict our attention to summative handicaps.

[4] Amann and Leininger [1] present two numerical examples of two-player all-pay auctions under asymmetric distributions. In their examples, and under some parameter values, the bidding distribution of one player "has an atom at 0", i.e., the player exits from the auction when his or her value is lower than some threshold.

[5] The existence of pure strategy equilibria in asymmetric auctions (or in games with incomplete information more generally) is shown using a variety of techniques by

a closed-form expression of the equilibrium strategy is sometimes unavailable, and therefore it is difficult to investigate problems with revenue or efficiency or the equilibrium allocation. [6] A study more closely related to the present analysis is Mares and Swinkels [13, 14] in that they consider a first-price auction with various handicap formulations. However, the motivation of their paper is not restricted to deriving the equilibrium strategy in a closed form, and they argue that the equilibrium strategy is usually complicated and so remains an open question. Our result then contrasts sharply with the complications found in many asymmetric auction studies.

Proposition 1 encompasses the key finding of our paper. This shows that the equilibrium strategy is written as a "piecewise affine transformation" of the equilibrium strategy in an all-pay auction without handicaps. Section 3.2 presents the central idea underlying this result where we argue that the optimality of the equilibrium strategy in an auction without handicaps directly proves the optimality of the equilibrium strategy in a handicap auction. Section 3.3 then discusses the allocation rule implemented in the equilibrium. In asymmetric auctions, it is rather obvious that inefficient allocation takes place with some positive probability. However, details of the allocation rule in the equilibrium are not straightforward. Here, we discuss that the shape of the hyperplane dividing the type space into each player's winning is "kinked".

Following this, Section 4 examines the uniform distribution case and

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Athey [3], Jackson and Swinkels [8], Lebrun [12], Maskin and Riley [17], and Reny [24].

[6] Maskin and Riley [16] examine asymmetric first-price auctions in detail. For some class of distributions, it is shown that equilibrium strategies in an asymmetric first-price auction can be explicitly derived (see Plum [23] and Cheng [6]). Marshall et al. [15] analyze the computation of equilibrium under the asymmetry of distributions in first-price auctions.

considers the problem of revenue comparison. Although "flat" handicaps appear to enhance competitive bids in the equilibrium, a change in one player's handicap affects other players' decisions in complicated ways, so it is not obvious if a decrease in one player's handicap always leads to an increase in revenue to the seller. In Section 4.1, we provide an example of a three-player case where an increase in one player's handicap increases revenue. We conclude in Section 5. The Appendix contains formal proofs of the propositions.

2 The Model

The basic structure of the all-pay auction with handicaps is as follows. There are N risk-neutral agents competing for a single prize. Agent i's value X_i is distributed over the interval $\mathcal{X} = [0, \bar{x}]$ according to an identical distribution function F with associated density function f.[7] We assume that agents' values are private and independently distributed.

In this model, each agent has a "handicap" of A_i that is common knowledge among all agents.[8] When each agent submits a sealed bid of b_i, the prize goes to the agent whose sum of the handicap and the bid is the maximum; that is, to agent i that satisfies:

$$b_i - A_i = \max\{b_1 - A_1, \dots, b_N - A_N\}.$$

[7]If we remove the identical distribution assumption, numerous complications arise and our result that the equilibrium with handicaps is a piecewise affine transformation of the equilibrium without handicaps is no longer valid.

[8]The model in the present paper is mathematically equivalent to a model where each agent i must pay a (nonrefundable) entry fee of A_i prior to the auction in order to be able to submit bids. We note that in order to establish the equivalence between the two models, we need to assume that agents who pay the entry fees submit a bid *before* observing whom else also participates in the auction. In this sense, interpreting A_i as an entry fee should be done on a contingent basis.

(If there is a tie, the prize goes to each winning agent with equal probability.) The bid, b_i, can be seen as the (irreversible) investment of efforts in contexts such as lobbying, bribery, contests, and patent races. The handicap, A_i, on the other hand, describes the disparity between the agents stemming from the accumulation of past efforts, hereditary advantage, or some discriminatory treatment by the sponsor of the contest. In the paper, we assume without loss of generality that:

$$0 = A_1 \leq A_2 \leq \cdots \leq A_N,$$

that is, Agent 1 is the most advantaged, Agent 2 is the second-most advantaged, and so on.

Given these bids and handicaps, the payoffs in the all-pay auction with handicaps are:

$$\Pi_i = \begin{cases} x_i - b_i & \text{if } b_i - A_i > \max_{j \neq i}(b_j - A_j) \\ -b_i & \text{if } b_i - A_i < \max_{j \neq i}(b_j - A_j). \end{cases}$$

A strategy for an agent i in this model is a function $\beta_i : [0, \bar{x}] \to \mathbb{R}_+$ that determines his bid for any value. For future use, we let $\bar{\beta}^k$ denote the symmetric equilibrium in the k-person all-pay auction *without* handicaps:[9]

$$\bar{\beta}^k(x) = \int_0^x y(k-1)F(y)^{k-2}f(y)dy. \tag{1}$$

We also let $\bar{\Pi}^k(x, z)$ denote an agent's (interim) expected payoff from bidding $\bar{\beta}^k(z)$ when his value is x, given that other bidders follow the strategy

[9]See, for instance, Krishna [10] and Milgrom [18].

$\bar{\beta}^k$ in the k-person all-pay auction without handicaps:

$$\bar{\Pi}^k(x,z) = F(z)^{k-1}x - \bar{\beta}^k(z). \tag{2}$$

As $\bar{\beta}^k$ constitutes an equilibrium, $\bar{\Pi}^k(x,z)$ satisfies:

$$\bar{\Pi}^k(x,x) \geq \bar{\Pi}^k(x,z) \quad \text{for all } x, z.$$

3 Equilibrium

3.1 Illustrative Examples

In this subsection, we provide two examples that illustrate the equilibrium behavior given in the main result below. Here we consider a model where there are three agents and the distribution F is uniform on an interval $[0,1]$.

Example 1. Suppose $A_2 = 1/4$ and $A_3 = 1/2$. Then:

$$\beta_1(x_1) = \begin{cases} 0 & \text{when } x_1 \in [0, z_2] \\ z_3\bar{\beta}^2(x_1) - C^2 & \text{when } x_1 \in (z_2, z_3] \\ \bar{\beta}^3(x_1) - C^3 & \text{when } x_1 \in (z_3, 1] \end{cases}$$

$$\beta_2(x_2) = \begin{cases} 0 & \text{when } x_2 \in [0, z_2] \\ z_3\bar{\beta}^2(x_2) - C^2 + A_2 & \text{when } x_2 \in (z_2, z_3] \\ \bar{\beta}^3(x_2) - C^3 + A_3 & \text{when } x_2 \in (z_3, 1] \end{cases}$$

$$\beta_3(x_3) = \begin{cases} 0 & \text{when } x_3 \in [0, z_3] \\ \bar{\beta}^3(x_3) - C^3 + A_3 & \text{when } x_3 \in (z_3, 1], \end{cases}$$

where $z_2 = \sqrt[6]{1/48} \approx 0.525$, $z_3 = \sqrt[3]{3/4} \approx 0.909$, $C^2 = 1/8$ and $C^3 = 1/4$,

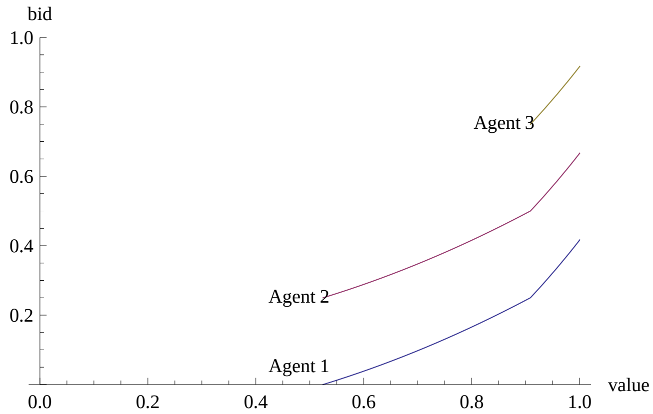

Figure 1: $A_2 = 1/4$, $A_3 = 1/2$

constitutes an equilibrium (Figure 1).

In the equilibrium, z_i is an "entry point" for agent i; that is, agent i makes positive bids only when his value is greater than z_i. In each interval $(z_k, z_{k+1}]$, each agent follows a strategy that is an affine transformation of the equilibrium strategy in k-person auction *without* handicaps. As shown later, the proof that such a set of strategies constitutes an equilibrium is also given in relation to the equilibrium without handicaps.

Example 2. Suppose $A_2 = 1/3$ and $A_3 = 2/3$. Then Agent 3 is "excluded" from the auction, and:

$$\beta_1(x_1) = \begin{cases} 0 & \text{when } x_1 \in [0, z_2] \\ \bar{\beta}^2(x_1) - C^2 & \text{when } x_1 \in (z_2, 1] \end{cases}$$

$$\beta_2(x_2) = \begin{cases} 0 & \text{when } x_2 \in [0, z_2] \\ \bar{\beta}^2(x_2) - C^2 + A_2 & \text{when } x_2 \in (z_2, 1] \end{cases}$$

$$\beta_3(x_3) = 0$$

where $z_2 = \sqrt{1/3} \approx 0.577$ and $C^2 = 1/6$ constitutes an equilibrium (Fig-

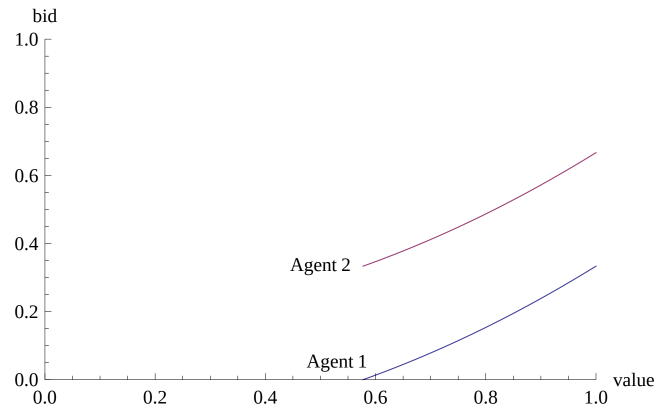

Figure 2: $A_2 = 1/3$, $A_3 = 2/3$ (Agent 3 is excluded)

ure 2).

3.2 Main Result

In this subsection, we provide the main result of the equilibrium analysis for the game with N agents and general distribution function F.

As illustrated in the examples, we are typically interested in the equilibrium of the following form:

$$\beta_i(x_i) \begin{cases} = 0 & \text{when } x_i \leq z_i^{\mathbf{A}} \\ > 0 & \text{when } x_i > z_i^{\mathbf{A}}, \end{cases}$$

where $z_i^{\mathbf{A}} \in [0, \bar{x}]$ is the "entry point" of agent i. In such an equilibrium (if any), agent i completely gives up winning when his value is no greater than $z_i^{\mathbf{A}}$, and makes positive bids only when $x_i > z_i^{\mathbf{A}}$. In general, $z_i^{\mathbf{A}}$ depends not only on his own A_i, but also on the vector of handicaps $\mathbf{A} = (A_1, \ldots, A_N)$.

In the auction with handicaps, it sometimes happens (as in the case of Agent 3 in Example 2) that an agent entirely abandons winning the auction when the agent's A_i is substantially large. In such a case, he always bids zero regardless of the value (that is, $\beta_i(x_i) = 0$ for all $x_i \in [0, \bar{x}]$). We describe this agent's strategy as $z_i^{\mathbf{A}} = \bar{x}$. We also let $z_{N+1}^{\mathbf{A}} \equiv \bar{x}$ for the sake of mathematical consistency.

Given the vector of entry points $\mathbf{z}^{\mathbf{A}} = (z_1^{\mathbf{A}}, \ldots, z_N^{\mathbf{A}})$, we let:

$$G^k(\mathbf{z}^{\mathbf{A}}) = \prod_{i=k+1}^{N+1} F(z_i^{\mathbf{A}})$$

and:[10]

$$C^k(\mathbf{z}^{\mathbf{A}}) = \sum_{\ell=2}^{k} \left[G^{\ell}(\mathbf{z}^{\mathbf{A}}) \bar{\beta}^{\ell}(z_{\ell}^{\mathbf{A}}) - G^{\ell-1}(\mathbf{z}^{\mathbf{A}}) \bar{\beta}^{\ell-1}(z_{\ell}^{\mathbf{A}}) \right] \tag{3}$$

We now have the following result.

Proposition 1. *Suppose that (i) $z_1^{\mathbf{A}}, \ldots, z_n^{\mathbf{A}} \in [0, \bar{x})$ is a solution to the system of equations:*

$$\begin{aligned} z_1^{\mathbf{A}} &= 0 \\ \{F(z_k^{\mathbf{A}})\}^{k-1} G^k(\mathbf{z}^{\mathbf{A}}) z_k^{\mathbf{A}} &= G^k(\mathbf{z}^{\mathbf{A}}) \bar{\beta}^k(z_k^{\mathbf{A}}) - C^k(\mathbf{z}^{\mathbf{A}}) + A_k \quad \text{for } k \in \{2, \ldots, n\} \end{aligned} \tag{4}$$

given that $z_{n+1}^{\mathbf{A}} = \cdots = z_{N+1}^{\mathbf{A}} = \bar{x}$, and (ii) the solution $\mathbf{z}^{\mathbf{A}}$ satisfies:

$$\bar{x} \leq \bar{\beta}^k(z_k^{\mathbf{A}}) - C^k(\mathbf{z}^{\mathbf{A}}) + A_k \tag{5}$$

[10] $G^k(\mathbf{z}^{\mathbf{A}})$ is regarded as the probability of the event that all agents $k+1, \ldots, N$ do not "enter" the auction. $C^k(\mathbf{z}^{\mathbf{A}})$ is determined in such a way that the equilibrium strategy is continuous everywhere except for each agent's entry point.

for $k = n+1, \ldots, N$. *Then:*

$$\beta_i(x) = \begin{cases} 0 & \text{when } x \in [0, z_i^{\mathbf{A}}] \\ G^k(\mathbf{z}^{\mathbf{A}})\bar{\beta}^k(x) - C^k(\mathbf{z}^{\mathbf{A}}) + A_i & \text{when } x \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}] \text{ and } i \leq k \end{cases} \tag{6}$$

constitutes a Bayesian–Nash equilibrium of the all-pay auction with handicaps $\mathbf{A}$.

Proof. See the Appendix. □

While the complete proof is given in the Appendix, we sketch here the idea underlying the result.

The main point of the result is that the equilibrium strategy of agent i with a value of $x_i > z_i^{\mathbf{A}}$ is an affine transformation of the equilibrium strategy in an all-pay auction without handicaps. In what follows, we generally argue that such strategies are the best responses to each other. Suppose that agent i has value $x_i \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$ ($k \geq i$), and other agents follow strategy β in (6). We consider what happens if agent i bids $\beta_i(y)$ where $y \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$.[11]

Agent i's probability of winning by bidding $\beta_i(y)$ is calculated as follows. The event that agent i wins against some agent $j \leq k$ is equivalent to the event that $X_j \leq y$, given we have from (6) that:

$$\beta_i(y) - A_i = \beta_j(y) - A_j$$

if $j \leq k$. The event that agent i wins against some agent $j > k$, on the other

[11] To complete the proof, we need to show that it is not beneficial for agent i to bid anything other than $\beta(x)$. See the Appendix for details.

hand, is equivalent to the event that $X_j \leq z_j^{\mathbf{A}}$, as we have from (5) that:

$$\beta_j(z_j^{\mathbf{A}}) - A_j = -A_j < 0 < \beta_i(y) - A_i \leq \beta_i(z_j^{\mathbf{A}}) - A_i = \lim_{x_j \downarrow z_j^{\mathbf{A}}} \beta_j(x_j) - A_j$$

if $j > k$.[12] Put together, we now have the probability of winning when agent i bids $\beta_i(y)$ ($y \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$) as:

$$q_i^k(y, \mathbf{z}^{\mathbf{A}}) = F(y)^{k-1} G^k(\mathbf{z}^{\mathbf{A}}) \quad \text{if } i \geq k. \tag{7}$$

We can then write agent i's expected payoff from bidding $\beta_i(y)$ when his value is x as follows:

$$\begin{aligned} \Pi_i(x,y) &\equiv q_i^k(y, \mathbf{z}^{\mathbf{A}})x - \beta_i(y) \\ &= G^k(\mathbf{z}^{\mathbf{A}})\bar{\Pi}^k(x,y) + C^k(\mathbf{z}^{\mathbf{A}}) - A_i, \end{aligned}$$

where $\bar{\Pi}^k$ is the corresponding expected payoff in a k-person auction without handicaps, as given in (2). Given $\bar{\Pi}^k$ satisfies $\bar{\Pi}^k(x,x) \geq \bar{\Pi}^k(x,y)$, we have:

$$\Pi_i(x,x) \geq G^k(\mathbf{z}^{\mathbf{A}})\bar{\Pi}^k(x,y) + C^k(\mathbf{z}^{\mathbf{A}}) - A_i = \Pi_i(x,y)$$

for all $x, y \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$. Thus, we have argued that agent i with a value of $x \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$ ($k \geq i$) cannot improve his payoff by bidding $\beta_i(y)$ ($y \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$).

In the Appendix, while we show that it is neither beneficial for agent i with a value of $x \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$ to make a bid $b \leq \beta_i(z_k^{\mathbf{A}})$ nor to make a bid $b > \beta_i(z_{k+1}^{\mathbf{A}})$, the spirit of the proof is identical to the above discussion.

[12] The argument here implicitly assumes that $\beta_i(x)$ is nondecreasing (and strictly increasing in $x \in (z_i^{\mathbf{A}}, \bar{x}]$). See Lemma 1 in the Appendix.

3.3 Allocation Rule

In this subsection, we provide a brief argument about the allocation rule implemented in the equilibrium given in Proposition 1.

Let $Q_i(\mathbf{x})$ be defined as the probability that i will win the auction in the equilibrium when agents' values are $\mathbf{x} = (x_1, \ldots, x_N)$. We then have the following:[13]

Proposition 2. *In the equilibrium of Proposition 1, for $x_i \in (z_k, z_{k+1})$:*

$$Q_i(\mathbf{x}) = \begin{cases} 1 & \text{if } x_i^{\mathbf{A}} > z_i^{\mathbf{A}}, x_i > \max_{j\in\{1,\ldots,k\}\setminus\{i\}} x_j, \\ & \quad \text{and } \max_{j\in\{k+1,\ldots,N\}}\{x_j - z_j^{\mathbf{A}}\} \leq 0 \\ 0 & \text{if } x_i^{\mathbf{A}} > z_i^{\mathbf{A}} \text{ and } \Big[x_i < \max_{j\in\{1,\ldots,k\}\setminus\{i\}} x_j \text{ or} \\ & \quad \max_{j\in\{k+1,\ldots,N\}}\{x_j - z_j^{\mathbf{A}}\} > 0\Big] \\ 0 & \text{if } x_i \leq z_i^{\mathbf{A}} \end{cases}$$

Figure 3 depicts the allocation rule when there are two agents and $A_2 > 0$. While we have a natural conjecture that the dividing line (the "Myerson Line" à la Mares and Swinkels [13]) is not below the diagonal, the precise shape of the line is not straightforward. In the first-price auction with handicaps, as studied in Mares and Swinkels, the line is smoother but likely difficult to derive in a closed form. In the numerical examples of the asymmetric all-pay auction studied by Amann and Leininger [1], the Myerson Line is also smooth. In the all-pay auction studied in the present paper, the line is kinked and can be explicitly derived using the agents' entry points $\mathbf{z}^{\mathbf{A}}$.

[13]Tiebreaking does not take place with a positive probability in the equilibrium, and hence is omitted in Proposition 2.

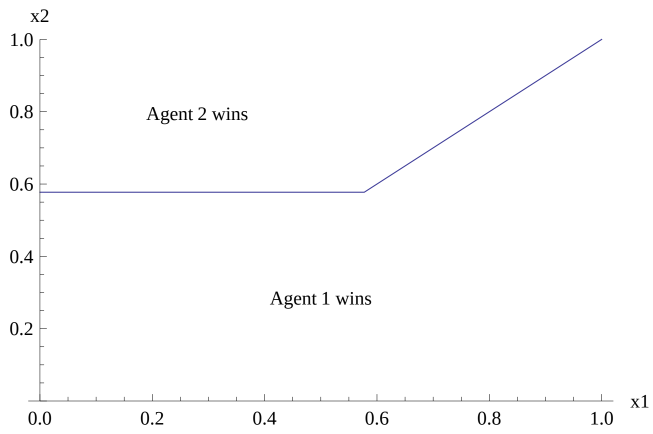

Figure 3: Myerson Line when $A_2 > 0$

Corollary. *With a positive probability, the all-pay auction with handicaps does not allocate the object efficiently.*

4 The Uniform Case

When the distribution, F, is uniform, we can analyze the agents' entry–exit decisions more explicitly than in Proposition 1.[14]

Proposition 3. *Suppose that the distribution F is uniform on an interval $[0,1]$. Given the handicaps $A_1, \ldots, A_N$:*

$$n^* = \max \left\{ n \;\middle|\; (n-1)A_n - \sum_{i=1}^{n-1} A_i < 1 \right\} \tag{8}$$

[14]The results in this section can be applied to the model where the distribution belongs to a subclass of the beta distribution that satisfies $F(x) = x^\alpha$ $(\alpha > 0)$. With such distributions, an agent's equilibrium bid is written as $cF(x)^{k-1}G^k(\mathbf{z}^\mathbf{A})x$ with some constant $c > 0$. This feature enables us to study the revenue problem in a closed form.

is the number of active agents in the equilibrium. That is:

$$z_i^{\mathbf{A}} \begin{cases} < 1 & \text{if } i \leq n^* \\ = 1 & \text{if } i > n^* \end{cases}$$

in the equilibrium. Moreover, $z_i^{\mathbf{A}}$ *for agents* $i \leq n^*$ *are recursively written as:*

$$z_i^{\mathbf{A}} = \left(\frac{B_i}{z_{i+1}^{\mathbf{A}}} \right)^{1/i} \tag{9}$$

where B_i *is defined as:*

$$B_i = (i-1)A_i - \sum_{j=1}^{i-1} A_j. \tag{10}$$

Proof. See the Appendix. □

We invite the reader to check that the values of $z_i^{\mathbf{A}}$ given in the examples in Section 3.1 can be calculated using the formula for Proposition 3.

Given the number of active agents n^* and entry decisions $(z_i^{\mathbf{A}})_{i \leq n^*}$, we can derive the affine transformations more explicitly.

Proposition 4. *Suppose that the distribution F is uniform on the interval* $[0, 1]$. *Given the handicaps* $\mathbf{A}$, *let* n^* *be the number of active agents as derived in (8). Then:*

$$\beta_i(x) = \begin{cases} 0 & \text{when } x \in [0, z_i^{\mathbf{A}}] \\ G^k \bar{\beta}^k(x) - C^k + A_i & \text{when } x \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}] \text{ and } i \leq k \end{cases}$$

constitutes an equilibrium of the all-pay auction with handicaps, where:

$$G^k = \begin{cases} \left(\prod_{\ell=k+1}^{n^*} (B_\ell)^{(1/\{\ell(\ell-1)\})}\right)^k & \text{if } k < n^* \\ 1 & \text{if } k = n^* \end{cases}$$

and

$$C^k = \frac{A_1 + \cdots + A_k}{k}.$$

Proof. See the Appendix. □

Our proof of the result that C^k is equal to the average of handicaps is strongly dependent on the assumption that the distribution is uniform, but it does suggest an interesting property of the equilibrium. For any given $x \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$, we have

$$\sum_{i=1}^{k} \beta_i(x) = k \cdot G^k \bar{\beta}^k(x),$$

so $\beta_i(x)$'s are symmetric around $G^k \bar{\beta}^k(x)$ for a given x.[15]

4.1 Revenue Comparison

In Section 3, we noted that the auction with handicaps results in inefficient allocation with a positive probability. It is not obvious, however, whether handicaps always have a tendency to decrease revenue to the seller. In this subsection, we provide a few results about the relationship between rev-

[15]Mares and Swinkels [14] note a conjecture that the agents' strategies "should move monotonically further apart" as the handicap grows in a two-person, first-price auction. In the equilibrium of the all-pay auction studied in the present analysis, the distance between the strategies is a constant, but we have a conjecture that the strategies should be symmetric (or distributed in a systematic way) around $G^k \bar{\beta}^k$ for the general distribution F.

enue and the handicaps using the explicitly derived formula of the equilibrium strategies when the distribution F is uniform.

Let $R_N^{\mathbf{A}}$ denote the ex ante expected revenue to the seller in the N-person all-pay auction with handicaps $\mathbf{A}$.

Proposition 5. *Suppose that the distribution F is uniform on the interval $[0,1]$. We then have:*

$$R_N^{\mathbf{A}} = \frac{N-1}{N+1} - \sum_{k=2}^{N} \frac{2}{k(k+1)} (z_k^{\mathbf{A}})^{k+1} \cdot z_{k+1}^{\mathbf{A}} \cdot \cdots \cdot z_N^{\mathbf{A}},$$

where $z_k^{\mathbf{A}}$ are as given in (9).

Proof. See the Appendix. □

It is not obvious how fluctuations in the agents' handicaps, $\mathbf{A}$, influence revenue as an increase in A_i has complex effects on all of the agents' entry decisions, $(z_i^{\mathbf{A}})$. The following example states that the problem is not straightforward.

Example 3. In some cases, an increase in an agent's handicap, A_i, increases revenue to the seller (Figure 4).

Figure 4 depicts the isoprofit curves of a three-person auction. When $A_3 > (A_2+1)/2$, Agent 3 is excluded, we therefore see $dR_3^{\mathbf{A}}/dA_3 = 0$. In the shaded area at the lower left, we can see that $dR_3^{\mathbf{A}}/dA_2 > 0$. This result is interpreted as follows: if A_3 is substantially large compared with A_2 (but $z_3^{\mathbf{A}} < 1$ is still satisfied), an increase in A_2 contributes to the enhancement of competitiveness between agents by setting Agent 3's mind at ease.

As to the handicap of the most disadvantaged agent, we have a monotonicity result.

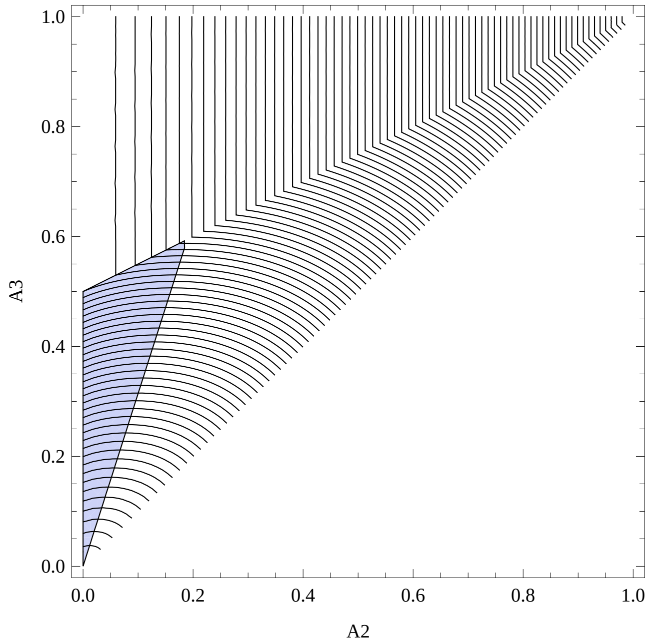

Figure 4: Isoprofit curves of three-person auction (in the shaded area, $dR_3^{\mathbf{A}}/dA_2$ is negative)

Proposition 6. *Suppose that the distribution F is uniform on* $[0,1]$*. Then we have:*

$$\frac{dR_N^{\mathbf{A}}}{dA_N} < 0$$

as long as $z_N^{\mathbf{A}} \in (0,1)$.

Proof. See the Appendix. □

It is a natural conjecture that Proposition 6 holds for general distribution F, but the proof given in the Appendix strongly depends on the assumption that the distribution is uniform. Hence, monotonicity in revenue with respect to handicaps remains an open question.

5 Concluding Remarks

In this paper, we have shown that the equilibrium strategy of the all-pay auction with handicaps is written in a closed form. Specifically, the strategy in each interval of values $(z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$ where exactly k agents are "active" is an affine transformation of the equilibrium strategy in the k-player all-pay auction *without* handicaps.

Technically, analyzing the equilibrium of asymmetric auctions (e.g., nonidentical distributions of values, risk aversion heterogeneity, and first-price auctions with handicaps) is relatively complicated. In most of these instances, and unlike symmetric auctions, a closed-form expression for the bidding strategies is unavailable and the allocations in the equilibrium are unclear. Such technical complications are resolved in the all-pay auction because of the feature that given an agent's value, the ex post amount of the bid in the equilibrium strategy is identical to the *interim* expected bid in the equilibrium.

Perhaps the most important question unanswered by the present analysis concerns the way in which the handicaps endowed in our model are formed in a more dynamic environment. For example, handicaps in a contest where agents compete for a final prize are sometimes a reflection of the agents' efforts in preceding periods. The general theoretical examination of these problems remains to be done.

Appendix

Lemma 1. *For each agent i, the strategy β_i given in (6) is continuous and strictly increasing in the interval $(z_i^{\mathbf{A}}, \bar{x}]$.*

Proof. It is already known that the equilibrium strategy in the all-pay auction *without* handicaps, given in (1), is continuous and strictly increasing. Now we show that β_i is continuous on every "junction" $z^{\mathbf{A}}_{i+1}, \ldots, z^{\mathbf{A}}_n$.

For some $k \geq i$, we have from (6) that:

$$\lim_{x \downarrow z^{\mathbf{A}}_{k+1}} \beta_i(x) = G^{k+1}(\mathbf{z^A})\bar{\beta}^{k+1}(z^{\mathbf{A}}_{k+1}) - C^{k+1}(\mathbf{z^A}) + A_i.$$

Using (3), we have:

$$\begin{aligned}\lim_{x \downarrow z^{\mathbf{A}}_{k+1}} \beta_i(x) &= G^{k}(\mathbf{z^A})\bar{\beta}^{k}(z^{\mathbf{A}}_{k+1}) - C^{k}(\mathbf{z^A}) + A_i \\ &= \beta_i(z^{\mathbf{A}}_{k+1}),\end{aligned}$$

thus β_i is continuous on the point $x = z^{\mathbf{A}}_{k+1}$ ($k \geq i$).

Given $\bar{\beta}^k$ is continuous and strictly increasing, we established that β_i is continuous and strictly increasing in the interval $(z^{\mathbf{A}}_i, \bar{x}]$. □

Given the vector of entry points $\mathbf{z^A}$, we define $\kappa(x)$ as follows:

$$\kappa(x) = \{k \mid x \in (z^{\mathbf{A}}_k, z^{\mathbf{A}}_{k+1}]\}$$

Let $\tilde{q}_i(x, \mathbf{z^A})$, $\tilde{\beta}_i(x)$ and $\tilde{\Pi}_i(x, y)$ be denoted respectively as:

$$\tilde{q}_i(x, \mathbf{z^A}) = \begin{cases} F(x)^{\kappa(x)-1} G^k(\mathbf{z^A}) & (i \geq k) \\ F(x)^{\kappa(x)} G^k(\mathbf{z^A}) / F(z^{\mathbf{A}}_i) & (i < k) \end{cases} \tag{11}$$

$$\tilde{\beta}_i(x) = G^{\kappa(x)}(\mathbf{z^A})\bar{\beta}^{\kappa(x)}(x) - C^{\kappa(x)}(\mathbf{z^A}) + A_i$$

$$\tilde{\Pi}_i(x, y) = \tilde{q}_i(y, \mathbf{z^A})x - \tilde{\beta}_i(y)$$

Note that $\tilde{\beta}_i([z^{\mathbf{A}}_2, \bar{x}]) = [A_i, \beta_i(\bar{x})]$ for every i.

Lemma 2. *For all $x \in [0, \bar{x}]$, $\tilde{\Pi}_i(x, y) - \tilde{\Pi}_i(x, y')$ is nondecreasing in x (nonincreasing in x) if $y > y'$ (if $y < y'$, respectively).*

Proof. We have:

$$\frac{d}{dx}\left\{\tilde{\Pi}(x, y) - \tilde{\Pi}(x, y')\right\} = \tilde{q}_i(y, \mathbf{z}^{\mathbf{A}}) - \tilde{q}_i(y', \mathbf{z}^{\mathbf{A}}).$$

From the definition given in (11), $q_i(y, \mathbf{z}^{\mathbf{A}})$ is nondecreasing in y. So $\tilde{\Pi}_i(x, y) - \tilde{\Pi}_i(x, y')$ is nondecreasing in x if $y > y'$. A similar observation holds for $y < y'$. □

Proof of Proposition 1.

Suppose that other agents $j \neq i$ follow the equilibrium strategy given in (6). First, note that it can never be optimal to choose a bid $b > \beta_i(\bar{x})$ as in that case agent i would certainly win and could do better by reducing his bid slightly. Similarly, it can never be optimal to choose a bid $b \in (0, A_i)$, as agent i would certainly lose and could do better by reducing his bid slightly. Accordingly, we need only consider bids $b \in \{0\} \cup [A_i, \beta_i(\bar{x})] = \{0\} \cup \tilde{\beta}_i([0, \bar{x}])$.

Given a value $x \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$ and $k \geq i$, suppose that agent i considers bidding $\beta_i(y)$ instead of $\beta_i(x)$. For $y \in (z_k^{\mathbf{A}}, z_{k+1}^{\mathbf{A}}]$ (i.e., in the "same" interval as x), it is straightforward from the definition that we have:

$$\tilde{\Pi}_i(x, x) \geq \tilde{\Pi}_i(x, y).$$

Thus, it is not beneficial to bid $\beta_i(y)$ instead of $\beta_i(x)$.

More specifically, we have for some $\ell < k$:

$$\tilde{\Pi}_i(z_{\ell+1}^{\mathbf{A}}, z_{\ell+1}^{\mathbf{A}}) \geq \tilde{\Pi}_i(z_{\ell+1}^{\mathbf{A}}, y)$$

for all $y \in (z^{\mathbf{A}}_{\ell}, z^{\mathbf{A}}_{\ell+1}]$. From Lemma 2, we obtain:

$$\tilde{\Pi}_i(x, z^{\mathbf{A}}_{\ell+1}) \geq \tilde{\Pi}_i(x, y)$$

for all $x \geq z^{\mathbf{A}}_{\ell+1}$, since $\tilde{\Pi}_i(x, z^{\mathbf{A}}_{\ell+1}) - \tilde{\Pi}_i(x, y)$ is nondecreasing in x. If $x \in (z^{\mathbf{A}}_{k}, z^{\mathbf{A}}_{k+1}]$, we obtain

$$\tilde{\Pi}_i(x, x) \geq \tilde{\Pi}_i(x, z^{\mathbf{A}}_{k}) \geq \cdots \geq \tilde{\Pi}_i(x, z^{\mathbf{A}}_{\ell+1}) \geq \tilde{\Pi}_i(x, y).$$

Then, we conclude that agent i with value x cannot benefit by bidding $b \in \{\tilde{\beta}_i(y) \mid y < x\}$. The same argument holds for the case of $y > x$. As $\Pi_i(x, x)$ is nondecreasing in x,

$$\Pi_i(x, x) \geq \Pi_i(z^{\mathbf{A}}_i, z^{\mathbf{A}}_i) = \Pi_i(z^{\mathbf{A}}_i, 0) = \Pi_i(x, 0) = 0$$

so agent i cannot benefit by bidding 0 when his value is greater than $z^{\mathbf{A}}_i$. Given $\tilde{\beta}_i([0, \bar{x}]) = [A_i, \beta_i(\bar{x})]$, we have argued that if all other agents are following the strategy β_j ($j \neq i$), agent i with a value of $x > z^{\mathbf{A}}_i$ cannot benefit by bidding $b \in \{0\} \cup [A_i, \beta_i(\bar{x})] \setminus \{\beta_i(x)\}$.

When agent i's value x is no greater than $z^{\mathbf{A}}_i$, then:

$$0 = \Pi_i(x, x) = \Pi_i(z^{\mathbf{A}}_i, z^{\mathbf{A}}_i) = \tilde{\Pi}_i(z^{\mathbf{A}}_i, z^{\mathbf{A}}_i) \geq \tilde{\Pi}_i(z^{\mathbf{A}}_i, y) \geq \tilde{\Pi}_i(x, y)$$

since $\tilde{\Pi}$ is nondecreasing in its first argument. Thus we have argued that if all other agents are following the strategy β_j ($j \neq i$), agent i with a value of $x \leq z^{\mathbf{A}}_i$ cannot benefit by bidding $b \in \{0\} \cup [A_i, \beta_i(\bar{x})] \setminus \{\beta_i(x)\}$. $\square$

Proof of Proposition 3.

First, note that if F is a uniform distribution on the interval $[0,1]$, the symmetric equilibrium strategy in a k-person all-pay auction *without* handicaps is:

$$\bar{\beta}^k(x) = \frac{k-1}{k}x^k \tag{12}$$

Thus, we can write:

$$\begin{aligned} C^k(\mathbf{z^A}) &= \sum_{\ell=2}^{k}\left[\frac{\ell-1}{\ell}(z_\ell^{\mathbf{A}})^\ell \prod_{m=\ell+1}^{N+1} z_m^{\mathbf{A}} - \frac{\ell-2}{\ell-1}(z_\ell^{\mathbf{A}})^{\ell-1}\prod_{m=\ell}^{N+1} z_m^{\mathbf{A}}\right] \\ &= \sum_{\ell=2}^{k}\left[\frac{\ell-1}{\ell}(z_\ell^{\mathbf{A}})^\ell \prod_{m=\ell+1}^{N+1} z_m^{\mathbf{A}} - \frac{\ell-2}{\ell-1}(z_\ell^{\mathbf{A}})^{\ell}\prod_{m=\ell+1}^{N+1} z_m^{\mathbf{A}}\right] \\ &= \sum_{\ell=2}^{k}\frac{1}{\ell(\ell-1)}(z_\ell^{\mathbf{A}})^\ell \prod_{m=\ell+1}^{N+1} z_m^{\mathbf{A}} \end{aligned} \tag{13}$$

From (4),

$$(z_k^{\mathbf{A}})^k \prod_{\ell=k+1}^{N+1} z_\ell^{\mathbf{A}} = \frac{k-1}{k}(z_k^{\mathbf{A}})^k \prod_{\ell=k+1}^{N+1} z_\ell^{\mathbf{A}} - \sum_{\ell=2}^{k}\frac{1}{\ell(\ell-1)}(z_\ell^{\mathbf{A}})^\ell \prod_{m=\ell+1}^{N+1} z_m^{\mathbf{A}} + A_k$$

We can show that:

$$(z_k^{\mathbf{A}})^k \prod_{\ell=k+1}^{N+1} z_\ell^{\mathbf{A}} = (k-1)A_k - \sum_{\ell=1}^{k-1} A_\ell \tag{14}$$

Given the right-hand side is increasing in k, there uniquely exists an n^* such that:

$$n^* = \max\left\{ n \;\middle|\; (n-1)A_n - \sum_{i=1}^{n-1} A_i < 1 \right\}$$

and n^* is the number of active agents in the equilibrium. For active agents $i = 1, \ldots, n^*$, we have:

$$z_i^{\mathbf{A}} = \left(\frac{B_i}{\prod_{j=i+1}^{n^*} z_j^{\mathbf{A}}} \right)^{1/i} = \frac{B_i^{1/i}}{\prod_{j=i+1}^{n^*} B_j^{1/\{j(j-1)\}}}$$

□

Proof of Proposition 4.

The derivation of G^k is straightforward.

From Equations (13) and (14), we have:

$$\begin{aligned}
C^k &= \sum_{\ell=2}^{k} \frac{1}{\ell(\ell-1)} \left\{ (\ell-1)A_\ell - \sum_{m=1}^{\ell-1} A_m \right\} \\
&= \sum_{\ell=2}^{k} \frac{A_\ell}{\ell} - \sum_{\ell=2}^{k} \sum_{m=1}^{\ell-1} \frac{A_m}{\ell(\ell-1)} \\
&= \sum_{m=2}^{k} \frac{A_m}{m} - \sum_{m=1}^{k-1} \sum_{\ell=m+1}^{k} \frac{A_m}{\ell(\ell-1)} \\
&= \sum_{m=2}^{k} \frac{A_m}{m} - \sum_{m=1}^{k-1} \left(\frac{1}{m} - \frac{1}{k-1} \right) A_m \\
&= \frac{1}{k} \sum_{m=1}^{k} A_m
\end{aligned}$$

□

Proof of Proposition 5.

Given G^k and C^k in Proposition 4:

$$E[\beta_i(x)] = \sum_{k=i}^{N} \int_{z_k}^{z_{k+1}} (G^k \bar{\beta}^k(x) - C^k + A_i) dx$$

$$= \sum_{k=i}^{N} \left(G^k \int_{z_k}^{z_{k+1}} \bar{\beta}^k(x)dx - C^k(z_{k+1}^{\mathbf{A}} - z_k^{\mathbf{A}}) \right) + A_i(1 - z_i^{\mathbf{A}})$$

$$= \sum_{k=i}^{N} \left(G^k \left[\frac{k-1}{k(k+1)} x^{k+1} \right]_{z_k^{\mathbf{A}}}^{z_{k+1}^{\mathbf{A}}} - C^k(z_{k+1}^{\mathbf{A}} - z_k^{\mathbf{A}}) \right) + A_i(1 - z_i^{\mathbf{A}})$$

As:

$$\begin{aligned}
\sum_{i=1}^{N} \sum_{k=i}^{N} C^k(z_{k+1}^{\mathbf{A}} - z_k^{\mathbf{A}}) &= \sum_{k=1}^{N} k \cdot C^k(z_{k+1}^{\mathbf{A}} - z_k^{\mathbf{A}}) \\
&= \sum_{k=1}^{N} \sum_{\ell=1}^{k} A_\ell(z_{k+1}^{\mathbf{A}} - z_k^{\mathbf{A}}) \\
&= \sum_{\ell=1}^{N} A_\ell \sum_{k=\ell}^{N} (z_{k+1}^{\mathbf{A}} - z_k^{\mathbf{A}}) \\
&= \sum_{i=1}^{N} A_i(1 - z_i^{\mathbf{A}})
\end{aligned}$$

we have:

$$\begin{aligned}
R_N^{\mathbf{A}} &= \sum_{i=1}^{N} E[\beta_i(x)] \\
&= \sum_{k=1}^{N} \frac{k-1}{k+1} \left[(z_{k+1}^{\mathbf{A}})^{k+1} - (z_k^{\mathbf{A}})^{k+1} \right] z_{k+1}^{\mathbf{A}} \cdot \dots \cdot z_N^{\mathbf{A}} \\
&= \frac{N-1}{N+1} + \sum_{k=2}^{N} \left(\frac{k-2}{k} - \frac{k-1}{k+1} \right) (z_k^{\mathbf{A}})^{k+1} \cdot z_{k+1}^{\mathbf{A}} \cdot \dots \cdot z_N^{\mathbf{A}} \\
&= \frac{N-1}{N+1} - \sum_{k=2}^{N} \frac{2}{k(k+1)} (z_k^{\mathbf{A}})^{k+1} \cdot z_{k+1}^{\mathbf{A}} \cdot \dots \cdot z_N^{\mathbf{A}}
\end{aligned}$$

□

Proof of Proposition 6

Let g_k denote the equilibrium gross expected profit of agent k with a value of $z_k^{\mathbf{A}}$ as follows:

$$g_k = (z_k^{\mathbf{A}})^k z_{k+1}^{\mathbf{A}} \dots z_N^{\mathbf{A}}.$$

Then:

$$\ln g_k = k \ln z_k^{\mathbf{A}} + \sum_{\ell=k+1}^{N} \ln z_\ell^{\mathbf{A}} \tag{15}$$

In particular,

$$\ln g_N = N \ln z_N^{\mathbf{A}},$$

hence

$$(N+1) \ln g_N = N \ln(g_N z_N^{\mathbf{A}}).$$

By taking the difference in (15),

$$\ln g_{k+1} - \ln g_k = (k+1) \ln z_{k+1}^{\mathbf{A}} - k \ln z - \ln z_{k+1}^{\mathbf{A}} = k(\ln z_{k+1}^{\mathbf{A}} - \ln z_k^{\mathbf{A}}),$$

Hence:

$$(k+1)(\ln g_{k+1} - \ln g_k) = k(\ln(g_{k+1} z_{k+1}^{\mathbf{A}}) - \ln(g_k z_k^{\mathbf{A}})).$$

Thus:

$$\begin{aligned}
\ln(g_N z_N^{\mathbf{A}}) - \ln(g_k z_k^{\mathbf{A}}) &= \sum_{\ell=k+1}^{N} \frac{\ell}{\ell-1} (\ln g_\ell - \ln g_{\ell-1}) \\
&= \frac{N}{N-1} \ln g_N + \sum_{\ell=k+1}^{N-1} \frac{1}{\ell(\ell-1)} \ln g_\ell - \frac{k+1}{k} \ln g_k,
\end{aligned}$$

hence,

$$\begin{aligned}\ln(g_k z_k^{\mathbf{A}}) &= \ln(g_N z_N^{\mathbf{A}}) - \frac{N}{N-1}\ln g_N - \sum_{\ell=k+1}^{N-1}\frac{1}{\ell(\ell-1)}\ln g_\ell + \frac{k+1}{k}\ln g_k \\ &= \frac{k+1}{k}\ln g_k - \sum_{\ell=k+1}^{N}\frac{1}{\ell(\ell-1)}\ln g_\ell.\end{aligned}$$

Therefore:

$$\frac{\partial \ln(g_k z_k^{\mathbf{A}})}{\partial \ln g_N} = \begin{cases}\frac{N+1}{N} & \text{if } k = N \\ -\frac{1}{N(N-1)} & \text{if } k < N\end{cases}$$

Thus:

$$\frac{\partial \ln\left(\sum_{k=1}^{N-1}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}}\right)}{\partial \ln g_N} = \frac{\frac{\partial \sum_{k=1}^{N-1}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}}}{\partial \ln g_N}}{\sum_{k=1}^{N-1}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}}} = -\frac{1}{N(N-1)},$$

hence,

$$\frac{\partial \sum_{k=1}^{N-1}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}}}{\partial \ln g_N} = -\frac{1}{N(N-1)}\sum_{k=1}^{N-1}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}},$$

and similarly:

$$\begin{aligned}\frac{\partial \frac{2}{N(N+1)}g_N z_N^{\mathbf{A}}}{\partial \ln g_N} &= \frac{N+1}{N}\frac{2}{N(N+1)}g_N z_N^{\mathbf{A}} \\ &= \frac{1}{N(N-1)}\left(\frac{2N}{N+1} - \frac{2}{N(N+1)}\right)g_N z_N^{\mathbf{A}}.\end{aligned}$$

Therefore:

$$\frac{\partial \sum_{k=1}^{N}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}}}{\partial \ln g_N} = \frac{1}{N(N-1)}\left(\frac{2N}{N+1}g_N z_N^{\mathbf{A}} - \sum_{k=1}^{N}\frac{2}{k(k+1)}g_k z_k^{\mathbf{A}}\right)$$

$$= \frac{1}{N(N-1)} \sum_{k=1}^{N} \frac{2}{k(k+1)} (g_N z_N^{\mathbf{A}} - g_k z_k^{\mathbf{A}})$$

$$> 0,$$

as long as $z_N^{\mathbf{A}} > 0$.

Thus:

$$\frac{dE[R]}{dA_N} = -\frac{\partial \ln g_N}{\partial A_N} \frac{\partial \sum_{k=1}^{N} \frac{2}{k(k+1)} g_k z_k^{\mathbf{A}}}{\partial \ln g_N} < 0.$$

□

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