On Decreasing Marginal Impatience

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Abstract

In the theory of endogenous time preference, one of the most controversial assumptions is that the degree of impatience is marginally increasing in wealth. We examine the implications of an empirically more relevant specification that time preference exhibits decreasing marginal impatience (DMI). With DMI, there are multiple steady-state nonsatiated and satiated equilibria. In a constant interest rate economy, the nonsatiated steady-state point is necessarily unstable, which results in the rich accumulating wealth up to a satiated level, and the poor shrinking toward zero consumption. In a capital economy with a decreasing returns technology, both of the nonsatiated and satiated steady-state points can be saddle-point stable. The model is used to examine the effects of capital taxation and government spending on capital accumulation.

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1 Introduction

When the degree of impatience, measured by the rate of time preference, is decreasing in wealth, the wealthier are more patient and, ceteris paribus, become even wealthier over time. Decreasing marginal impatience (thereafter DMI) is thus by nature destabilizing. To avoid the resulting analytical difficulty, the theory of endogenous time preference has usually assumed increasing marginal impatience (thereafter IMI) (e.g., Uzawa (1968) and Epstein and Hynes (1983)) and, irrespective of the large amount of research on time preference, limited research has so far been conducted on the theoretical or economic implications of DMI. However, Svensson and Razin (1983) proposed that unstable equilibrium time paths of economies cannot be ruled out a priori. In fact, existing empirical studies, using either actual economic data (e.g., Lawrance (1991) and Samwick (1998)) or using experimental data (e.g., Harrison et al. (2002) and Ikeda et al. (2005)), commonly and strongly support the validity of DMI. It is of critical importance to determine the implications of DMI.1

The purposes of the present paper are: (i) to examine the workings of a dynamic economic system under DMI; and (ii) to consider the policy implications. To do so, the optimal consumer behavior is examined first by using a model of a constant interest rate economy, and then by using a capital economy model with variable interest rates.

With standard regularity conditions such as concavity, we first show that under DMI there are multiple steady-state nonsatiated and satiated equilibria. The presence of the satiated steady-state equilibrium allows us to consider the unstable optimal dynamics around usual nonsatiated steady-state points. In a constant interest rate economy, the nonsatiated steady-state point is shown to be necessarily unstable, which leads the rich to accumulate wealth up to a satiated level, and the poor to shrink toward zero consumption.

In the neoclassical production economy with decreasing returns to capital,

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1 As for the literature of a different interest, Becker and Mulligan (1997) discuss decreasing marginal impatience by using a ‘future-oriented’ capital model, where the wealthier invest more ‘future-oriented’ capital to obtain lower time preference. By using a hyperbolic discounting model, Barro (1999) derives the possibility that time preference is decreasing in consumption during transition. Although it could be regarded as an observationally equivalent phenomenon to ours, the model is time inconsistent without any commitment tools.
the usual modified golden-rule steady-state point, as well as the satiated one, can be saddle-point stable. The model is used to examine the effects of capital taxation and government spending on capital accumulation.

An increase in capital taxes raises the long-run after-tax interest rate, so that, in contrast to what Epstein and Hynes (1981) show under IMI, the resultant decreases in steady-state capital, consumption, and hence welfare are larger than in the case of constant time preference. When time preference exhibits DMI, a decrease in the capital stock caused by capital taxation makes consumers less patient, which raises the long-run interest rate irrespective of the capital tax increase, and thereby causes a further reduction in the capital stock. An increase in government spending, i.e., a negative income shock, under DMI raises the long-run interest rate and leads to a reduction in the long-run capital stock. This is sharply contrasted again with the implausible result under IMI that a negative income shock enlarges the capital stock in the long run.

Das (2003), in an important contribution to a similar issue, also shows that DMI is compatible with saddle-point stability in the neoclassical exogenous growth model. Our contribution differs in several points. First, unlike in his model, we impose standard regularity conditions on consumer preferences, which enables us to discuss both the cases of DMI and IMI under the same setting. Second, with the regularity conditions, we obtain satiated steady-state solutions, which allow us to analyze optimal dynamics even if the usual nonsatiated steady-state point is unstable. Third, this paper works out the macroeconomic policy implications of DMI.

The rest of the paper is organized as follows. Section 2 considers the effects of DMI by using a simple intertemporal utility-maximization problem with a constant interest rate. In Section 3, the analysis is extended to the neoclassical model with capital accumulation to examine the effects of capital taxation and of government spending. Section 4 concludes the paper.

2 The effects of decreasing marginal impatience

To examine the effects of DMI, consider an infinitely lived consumer who maximizes lifetime utility by choosing the time profiles of consumption \( \{c(t)\}_{t=0}^{\infty} \) and total asset holdings \( \{a(t)\}_{t=0}^{\infty} \). His or her problem is specified in a simple
form as

\[ \max \int_{0}^{\infty} u(c(t)) \exp(-\Delta(t)) \, dt, \]

subject to:

\[ \dot{a}(t) = ra(t) - c(t), \quad a(0) = a_0 \, \text{(constant)}, \quad a(t) \geq 0, \]

\[ \dot{\Delta}(t) = \delta(c(t)), \quad \Delta(0) = 0, \]

where a dot represents the time derivative; \( u(c) \) represents the felicity function; and \( \Delta \) denotes a cumulative discount rate with the instantaneous discount rate being given by \( \delta(c) : \Delta(t) = \int_{0}^{t} \delta(c(\tau)) \, d\tau \).

Functions \( u \) and \( \delta \) are twice continuously differentiable. The rate of interest \( r \) is assumed to be constant in this case.\(^2\)

As in the literature (e.g., Epstein (1987a) and Obstfeld (1990)), the consumer preferences are assumed to satisfy the following standard regularity conditions:

(A1) \( u < 0; \)

(A2) \( u \) and \( \delta \) are concave; and

(A3) \(-u\) is log-convex.

With these conditions, the optimal solution is sufficiently given by the standard first-order conditions.

As is well known (and also will be shown later by (13)), the degree of impatience is marginally increasing or decreasing in wealth as \( \delta \) is increasing or decreasing in consumption. Note that what is required to the discount function \( \delta \) by regularity conditions (A1) - (A3) is just concavity;\(^3\) and they are not related to whether the subjective discount rate \( \delta \) is increasing or decreasing. Most generally, it can be hump shaped, where impatience is

\(^2\)Asset holding \( a \) includes both financial wealth and human capital. We can rewrite the positivity condition for \( a \) as the no-Ponzi game condition to bond holdings.

\(^3\)The regularity conditions, including the concavity of \( \delta \), ensure the concavity of the lifetime utility function. By using a two-period model, Obstfeld (1990) discusses this point intuitively. For a detailed proof in the infinite horizon case, see Hirose and Ikeda (2004a).
increasing for smaller consumption at some critical quantity whereas it is decreasing for larger quantities of consumption.\textsuperscript{4}

**Remark 1:** Several researchers (e.g., Fukao and Hamada (1991) and Jafarey and Park (1998)) consider U-shaped nonmonotonic discount-rate functions. Their discount-rate functions do not satisfy the regularity conditions (A1) - (A3), implying that the usual first-order conditions may not give the optimality conditions.

Let us focus on the case of DMI by additionally assuming:

(A4) impatience is marginally decreasing: \( \delta_c(c) < 0 \); and

(A5) the upper bound \( \bar{\delta} (\equiv \delta(0)) \) of \( \delta(c) \) satisfies \( \bar{\delta} > r \).\textsuperscript{5}

where \( \delta_c(c) \) represents \( d\delta(c)/dc \).

Assumption (A5) is related to the existence of a steady-state solution. From (A2), (A4), and (A5), there uniquely exists \( c^* > 0 \), such that

\[
\delta(c^*) = r. 
\]

The bottom panel of Figure 1 depicts the resulting typical \( \delta(c) \)-schedule.

With increasing marginal impatience, a steady-state optimal solution would be uniquely given by (4). In the case of DMI, there is another steady-state solution, which is characterized by satiation. This takes place because, due to the negativity of utility ((A1)), an increase in consumption has a negative impact on the lifetime utility level by lowering the discount rate when the consumption level is large enough.

More explicitly, let \( c' \) be any consumption level such that \( \delta(c') > 0 \). From concavity ((A2)), \( \delta \) satisfies that for \( c > c' - \delta(c')/\delta_c(c') \),

\[
\delta(c) < \delta(c') + \delta_c(c') (c - c') < 0 , 
\]

implying that, as illustrated by the bottom panel of Figure 1, \( \delta \) is negative for consumption levels that are higher than a finite critical value \( \bar{c} \):

\[
\delta(c) \lessgtr 0 \text{ as } c \lessgtr \bar{c} .
\]

\textsuperscript{4}Das (2003) proposes another set of regularity conditions under which \( \delta \) should be decreasing in \( c \). In this case we cannot deal with the mixed case in which impatience can be marginally increasing or decreasing depending on the consumption level.

\textsuperscript{5}From assumptions (A2) and (A4), \( \delta(c) \) necessarily has the upper bound at \( c = 0 \).
If the lifetime utility generated by a stationary consumption path \( c(t) = c \) is denoted by \( U(c) \): 
\[
U(c) \equiv \int_0^\infty u(c) \exp(-\delta(c) t) dt,
\]
it thus satisfies

(5) 
\[
U(c) = \begin{cases} 
  u(c) / \delta(c), & \text{for } c < \bar{c} \\
  -\infty, & \text{for } c \geq \bar{c}
\end{cases},
\]

implying that, as illustrated by the \( U(c) \)-schedule in the top panel of Figure 1, there exists a satiated consumption level \( c^{**} \in [0, \bar{c}) \) such that

(6) 
\[
c^{**} = \arg\max_{c \geq 0} U(c).
\]

As for the satiated consumption level \( c^{**} \), we assume the following:

(A6) \( c^{**} \) is unique;

(A7) for \( c < c^{**} \), \( U(c) \) is strictly increasing; and

(A8) \( c^{**} > c^* \), where \( c^* \) is given in (A4).

Assumptions (A6) and (A7) are made simply for analytical simplicity. If assumption (A8) were not satisfied, the steady-state solution \( c^* \), defined by (A4), would not be optimal when \( c^* \) is strictly larger than the satiation level \( c^{**} \). The top panel of Figure 1 depicts the resulting typical \( U(c) \)-schedule.

The regularity conditions (A1) - (A3) enable us to apply the usual maximization procedures. Letting \( \lambda \) and \( \phi \) represent the current-value shadow prices for savings and the discount factor \( \Omega = \exp(-\Delta) \), respectively, the optimal conditions to maximize the lifetime utility function (1) are given by

(7) 
\[
\xi_c(c, \phi) = \lambda,
\]

(8) 
\[
\dot{\lambda} = (\delta(c) - r) \lambda,
\]

(9) 
\[
\dot{\phi} = -\xi(c, \phi),
\]

(10) 
\[
\lim_{t \to \infty} \exp(-\Delta(t)) \lambda(t) a(t) = 0,
\]

(11) 
\[
\lim_{t \to \infty} \exp(-\Delta(t)) \phi(t) \Delta(t) = 0,
\]
where $\xi$ represents the generating function:

$$\xi(c, \phi) = u(c) - \phi \delta(c).$$

Two points are noteworthy. First, the optimal $\phi(t)$ equals the lifetime utility obtained from the optimal consumption stream after time $t$, as can be seen by solving differential equation (9) under the transversality condition (11), and thus $\phi < 0$ from (A1). Second, $\xi_c(c, \phi) = u_c(c) - \phi \delta_c(c)$ represents the current-value marginal utility of $c$. Since $\frac{d(u(c)/\delta(c))}{dc} = (u_c - (u/\delta) \delta_c)/\delta = \xi_c(c, u/\delta)/\delta$, assumption (A7) implies

$$\xi_c(c, u/\delta) > 0 \text{ for } c < c^*, \quad 0 \text{ for } c = c^*,$$

meaning that the marginal utility evaluated at a stationary consumption time path $c(t) = c$ is strictly positive only when the consumption quantity is less than the satiation level, at which the marginal utility falls to zero. From assumption (A1) and $\phi < 0$, we obtain $\xi_{cc}(c, \phi) < 0$.

By substituting successively equations (7) and (9) into (8), we obtain the optimal consumption dynamics as

$$\dot{c} = -\frac{\xi_c(c, \phi)}{\xi_{cc}(c, \phi)}(r - \delta(c)) - \frac{\xi(c, \phi)}{\xi_{cc}(c, \phi)} \delta_c(c). \quad (12)$$

When $\xi_c > 0$, the rate of time preference $\rho$ can be defined as $\rho = -\frac{d \ln \Gamma(t)}{dt}|_{t=0}$, where $\Gamma \equiv \xi_c(c, \phi) \exp(-\Delta)$ represents the present value marginal utility of $c$, i.e.,

$$\rho(c, \phi) = \delta(c) - \frac{\xi(c, \phi)}{\xi_c(c, \phi)} \delta_c(c), \quad (13)$$

with which equation (12) can be rewritten in the usual form as

$$\dot{c} = -\frac{\xi_c(c, \phi)}{\xi_{cc}(c, \phi)}(r - \rho(c, \phi)). \quad (14)$$

The optimal time path for $(c, \phi, a)$ must be jointly generated by equations (2), (9), and (12) (or (14)). There are two steady-state optimal solutions: the (nonsatiated) steady-state solution $E^*$ and the satiated steady-state solution $E^{**}$, which are defined, respectively, by:

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6For the proof, see Hirose and Ikeda (2004a).
Nonsatiated steady-state solution $E^*$: $(c^*, \phi^*, a^*)$;

$$\delta(c^*) = r, \quad \phi^* = \frac{u(c^*)}{\delta(c^*)}, \quad ra^* = c^*, \quad \text{and} \quad \xi_c(c^*, \phi^*) > 0. \quad (15)$$

Satiated steady-state solution $E^{**}$: $(c^{**}, \phi^{**}, a^{**})$;

$$\xi_c(c^{**}, \phi^{**}) = u_c(c^{**}) - \phi^{**} \delta_c(c^{**}) = 0, \quad \phi^{**} = \frac{u(c^{**})}{\delta(c^{**})}, \quad \text{and} \quad ra^{**} = c^{**}. \quad (16)$$

Solution $E^*$ is characterized by the consumption level $c^*$ in (4) and a strictly positive $\xi_c$; and the satiated steady-state solution $E^{**}$ by the satiated consumption level $c^{**}$ and $\xi_c = 0$. By assumptions, the steady-state solutions $E^*$ and $E^{**}$ both uniquely exist.\(^7\)

Example 1: For a simple example, specify functions $u(c)$ and $\delta(c)$ in quadratic forms as

$$u(c) = -\left(\frac{100}{121}\right)(c - 11)^2, \quad (17)$$

$$\delta(c) = -\frac{0.1}{100}c^2 + 0.1, \quad (18)$$

respectively. These functions can be shown easily to satisfy the regularity conditions (A1) through (A5) for $c \in (0, 11)$.\(^9\) The critical consumption level $\bar{c}$ that satisfies $\delta(\bar{c}) = 0$ equals 10. The resulting $U(c)$-function (5) satisfies assumptions (A6) and (A7), where the satiated consumption level $c^{**}$ is uniquely given by 9.091. The nonsatiated steady-state equilibrium $E^*$ uniquely exists if $r \in [0.018, 0.1]$.\(^10\) For example, when $r = 0.05$, the steady-state consumption $c^*$ amounts to 7.071, which is smaller than satiated level 9.091, as desired by (A8).

\(^7\)This shows that, even if the felicity function is strictly increasing, satiation can arise under intertemporally nonseparable preferences. See Ryder and Heal (1973), which shows that habit formation can produce satiated steady-state optimal solutions.

\(^8\)Satiation has been reported in happiness studies; see, e.g., Leu et al. (1997) and Tsutsui et al. (2005).

\(^9\)Function (17) can be respecified to satisfy the regularity conditions for all $c \geq 0$ by modifying arbitrarily the graph for $c \geq 11$.

\(^10\)The exact value of the lower bound is 0.017355\ldots.
By analyzing the local dynamics around the two steady-state points, we can show that nonsatiated steady-state point $E^*$ is unstable whereas satiated point $E^{**}$ is saddle-point stable (see Appendix A).

**Proposition 1:**

Under assumptions (A1)-(A8), the optimization problem with DMI, given by (1) - (3), has two steady-state optimal points: a nonsatiated point $E^*$, which is unstable, and a satiated point $E^{**}$, which is saddle-point stable.

The resulting optimal consumption dynamics are depicted by Figure 2, where the top panel illustrates the $(c, a)$-dynamics and the bottom depicts $(c, \phi)$. In the bottom panel, we illustrate two $\dot{c} = 0$ schedules. One is defined by $\rho(c, \phi) = r$, and is locally flat at $E^*$. The other is upward sloping at $E^{**}$. Schedule $\dot{\phi} = 0$ illustrates the relation $\xi(c, \phi) = 0$. Since zero $\xi$ means $\phi = u(c)/\delta(c)$, the schedule is exactly the same as depicted in the top panel of Figure 1. Steady-state points $E^*(c^*, \phi^*)$ and $E^{**}(c^{**}, \phi^{**})$ are determined at the intersections of the $\dot{\phi} = 0$ schedule and the two $\dot{c} = 0$ schedules. By assumption, the $\dot{\phi} = 0$ schedule has a unique peak at $E^{**}$. The optimal consumption dynamics are indicated by arrows. Point $E^*$ is depicted as unstable whereas $E^{**}$ is saddle-point stable. Given an initial value $a_0$ of wealth holding, optimal consumption $c(0)$ is determined on the optimal trajectory in the $(c, a)$-plane. The $(c, \phi)$-dynamics are then generated on the arrowed trajectory in the $(c, \phi)$-plane.

The optimal consumption dynamics depend crucially on $a_0$. Suppose first that $0 < a_0 < a^*$. Then, $c(0)$ is determined as smaller than $c^*$, and thereafter $c(t)$ and hence $a(t)$ implode over time. Consider, instead, the case that $a^* < a_0 < a^{**}$. In this case, $c(0)$ exceeds $c^*$, but falls short of $c^{**}$. In the interim run, $c(t)$ grows gradually toward the satiation level $c^{**}$. Finally, note that, when $a_0 > a^{**}$, a larger $c(t)$ than $c^{**}$ would generate negative marginal utility. The optimal solution is thus not to choose greater consumption levels than $c^{**}$ even though they are feasible, but to keep $c(t)$ equal to the satiation level $c^{**}$.

As seen from the above discussions, DMI in a constant interest rate economy leads poor consumers to decumulate wealth toward zero and rich consumers to accumulate up to the satiated level. Note that the attained long-run consumption levels, zero or the satiation level, are insensitive to any income shocks, e.g., those in $a_0$ and/or $r$. With constant interest rates, the DMI model thus may not be suitable for analyzing the long-run effects of
policy changes.
This, however, does not imply that we cannot analyze any models of DMI. We can consider well-behaved models with DMI by introducing some stabilizing decreasing return properties into the production technology and/or preferences. We next incorporate capital accumulation with a usual decreasing return technology.

3 Decreasing marginal impatience and capital accumulation

3.1 The neoclassical model
Consider a stylized neoclassical model: there are two production factors, labor and capital; there is a single multi-purpose commodity; it is produced by using a constant-to-scale technology \( F \); and firms are competitive. Consumers supply inelastically one unit of labor at each instant. Their preferences are specified just as in the previous section. In particular, we assume DMI, \( \delta_c(c) < 0 \). The government spends \( g \) by levying capital taxes \( \tau \) and lump-sum taxes. Letting \( k \) represent the capital–labor ratio and \( f \) a per capita production function satisfying \( f_k > 0 \), \( f_{kk} < 0 \), and the Inada conditions, we can easily obtain a reduced dynamic system as follows:

\[
\dot{c} = -\frac{\xi_c(c, \phi)}{\xi_{cc}(c, \phi)} ((1 - \tau) f_k(k) - \delta(c)) - \frac{\xi(c, \phi)}{\xi_{cc}(c, \phi)} \delta_c(c),
\]

(19)

\[
\dot{\phi} = -\xi(c, \phi),
\]

\[
\dot{k} = f(k) - c - g.
\]

The solution of this system that satisfies the transversality condition is sufficiently optimal. When \( \xi_c(c, \phi) > 0 \), the first equation can be rewritten as

\[
\dot{c} = -\frac{\xi_c(c, \phi)}{\xi_{cc}(c, \phi)} ((1 - \tau) f_k(k) - \rho(c, \phi)),
\]

where the rate of time preference \( \rho(c, \phi) \) is given by (13).

As in the previous section, we consider two steady-state equilibria: (i) the nonsatiated, modified golden-rule steady-state equilibrium \( E^*(c^*, k^*) \) such
that

\[(20) \quad \delta (c^*) = (1 - \tau) f_k (k^*), \]

\[(21) \quad f (k^*) = c^* + g, , \]

and (ii) the satiated steady-state equilibrium \( E^{**} (c^{**}, k^{**}) \), where \( c^{**} \) is the satiated consumption level (6); and \( k^{**} \) is given by \( f (k^{**}) = c^{**} + g \) as in (21).

By analyzing the local dynamics around these two steady-state points, we can show that the nonsatiated steady-state point \( E^* \), as well as the satiated one \( E^{**} \), can be saddle-point stable:

**Lemma 1:**

1. Nonsatiated steady-state point \( E^* \) is saddle-point stable if and only if:

\[(22) \quad f_k (k^*) < (1 - \tau) f_{kk} (k^*) / \delta_c (c^*). \]

2. Satiated steady-state point \( E^{**} \) is saddle-point stable if and only if:

\[(23) \quad \delta (c^{**}) < (1 - \tau) f_k (k^{**}). \]

**Proof.** See Appendix B. ■

Figures 3 (a) and (b) demonstrate the determination and the stability property of steady-state points \( E^* \) and \( E^{**} \), where the relations (20) and (21) are depicted as upward sloping schedules. With the satiated consumption level \( c^{**} \) being determined from (6), satiated steady-state point \( E^{**} \) is determined on the (21) schedule. Nonsatiated steady-state point \( E^* \) is given at the intersection of the two schedules if it exists on the left side of \( E^{**} \). For point \( E^* \) to be saddle-point stable, condition (22) in lemma 1 requires that the gradient of the (21) schedule at \( E^* \) (the left-hand side of (22)) be smaller than that of (20) (the right-hand side). From (23), point \( E^{**} \) is saddle-point stable if and only if it is located above the (20) schedule.
Note that $\delta(0) = \bar{\delta}(\prec \infty)$ whereas, from the Inada condition, we have $\lim_{k \to 0} f_k(k) = \infty$. Schedule (20) thus intersects the horizontal axis at a positive $k$. Since schedule (21) goes through the origin when $g$ is assumed to equal zero for brevity, this implies that the relative magnitudes of $\delta(c^{**})$ and $(1 - \tau)f_k(k^{**})$, i.e., whether satiated steady-state point $E^{**}$ is located above or below schedule (20), has critical implications for the existence and stability of the nonsatiated and satiated steady-state points, as can be summarized as follows:\footnote{In the case that $g > 0$, Proposition 2 remains valid as far as $(1 - \tau)f_k(f^{-1}(g)) > \bar{\delta}$.}

**Proposition 2:**

1. Suppose that $\delta(c^{**}) > (1 - \tau)f_k(k^{**})$. Then,
   
   (a) there necessarily exist an odd number of nonsatiated steady-state points, which are alternatively saddle-point stable and unstable, i.e., the first one is saddle-point stable, the second is unstable, $\ldots$, and the last saddle-point stable; and
   
   (b) the satiated steady-state point is unstable.

2. Suppose that $\delta(c^{**}) < (1 - \tau)f_k(k^{**})$. Then,
   
   (a) if there exists a $k < k^{**}$ such that $\delta(f(k)) \geq (1 - \tau)f_k(k)$, there exist an even number of steady-state points, which are alternatively saddle-point stable and unstable, i.e., the first one is saddle-point stable, the second is unstable $\ldots$, and the last is unstable. Otherwise there exists a non-nonsatiated steady-state point; and
   
   (b) The satiated steady-state point is saddle-point stable.

Figure 3 (a) depicts the case that $\delta(c^{**}) > f_k(k^{**})$ by assuming that the nonsatiated steady-state point $E^*$ is unique. From (22), point $E^*$ is saddle-point stable whereas, from (23), satiated steady-state point $E^{**}$ is unstable, as stated by Proposition 2 (1). Insofar as the initial capital stock $k_0$ lies below the satiated stock level $k^{**}$, the economy monotonically converges to the nonsatiated steady-state point $E^*$. Figure 3 (b) illustrates the case that $\delta(c^{**}) < f_k(k^{**})$, where two nonsatiated steady-state points are assumed.
to exist. As implied by statement 2 (a) in Proposition 2, the first point $E^*_1$ is saddle-point stable whereas the second point $E^*_2$ is unstable. Point $E^{**}$ is saddle-point stable. When $k_0$ lies below $k^*_2$, the economy gradually approaches nonsatiated steady-state point $E^*_1$, whereas a higher $k_0$ than $k^*_2$ is followed by monotonic convergence toward satiated steady-state point $E^{**}$.

**Example 2:** As in Example 1, we specify functions $u(c)$ and $\delta(c)$ in quadratic forms as (17) and (18), respectively, and the production function as

$$f(k) = Ak^{0.3}, A > 0.$$ 

Assume that $g = 0$ and $\tau = 0$. These functions satisfy all the regularity conditions. The satiated consumption level is obtained as $c^{**} = 9.091$. The existence of the nonsatiated steady-state equilibrium $E^*$ depends on the value of total factor productivity $A$. From Proposition 2, there exists a nonsatiated equilibrium point if and only if for some $k < k^{**}$, $\delta(f(k)) \geq (1 - \tau)f_k(k)$ and hence $\delta(Ak^{0.3}) \geq 0.3Ak^{-0.7}$. With a too large $A$, however, the capital productivity $0.3Ak^{-0.7}$ remains larger than $\delta(Ak^{0.3})$ for all $k < k^{**}$. For the existence of the nonsatiated steady-state equilibrium, $A$ must thus be smaller than some critical value, which can be computed as 2.302 in the present example. After tedious computation, we can show the following relations.

1. For $A \in (0, 1.993]$, the nonsatiated steady-state equilibrium point $E^*$ uniquely exists, and is saddle-point stable, as in Figure 3 (a). For example, when $A = 1.5$, $(c^*, k^*)$ equals $(2.974, 9.786)$.

2. For $A \in [1.994, 2.301]$, there are two nonsatiated steady-state equilibria, one saddle-point stable and one unstable, as in Figure 3 (b). For example, when $A = 2.1$, $(c^*, k^*)$ is given by $(5.335, 22.373)$, which is saddle-point stable, and $(8.826, 119.811)$, which is unstable.

3. For $A \geq 2.302$, there is a non-nonsatiated equilibrium.

With a decreasing return technology, a model of DMI can thus be well behaved in the sense that there exists a nonsatiated steady-state point that is saddle-point stable. This enables us to consider the policy implications of DMI by conducting usual comparable statics.

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12 The exact critical value is 2.30160\ldots.

13 The exact value of the upper bound is 1.99397\ldots.
3.2 The effects of capital taxation

Let us examine the implications of DMI on capital taxation. As shown by Chamley (1981), when the rate of time preference is constant, capital shifts the entire burden of capital taxation in the long run because the long-run after-tax rate of return to capital must equal the constant time preference rate. The resulting reductions in the steady-state capital, consumption, and welfare are large. With endogenous time preference, Epstein and Hynes (1983) show that capital taxation reduces the steady-state capital stock, but not as much as it would under constant time preference, implying that the reductions in consumption and welfare are mitigated. However this result depends crucially on the assumption of marginal increasing impatience. With DMI, the result is drastically changed, as we shall show below.

Assume that the economy initially stays at a nonsatiated, saddle-point stable steady-state point. From (20) and (21), the steady-state capital stock \( k^* \) and the long-run after-tax rate of return to capital \( r^* \equiv (1 - \tau) f_k (k^*) \) are determined by

\[
(24) \quad r^* = (1 - \tau) f_k (k^*) = \delta (f (k^*) - g).
\]

As shown in Figure 4, \((1 - \tau) f_k (k^*)\) on the left-hand side can be depicted as a downward-sloping schedule in the \((r, k)\) plane. With DMI, the right-hand side, \(\delta (f (k^*) - g)\), can also be expressed by a downward-sloping schedule. The steady-state capital stock \(k^*\) and the long-run after-tax rate of return to capital \(r^*\) are given at the intersection, say point \(E_0\), of the two schedules. For the initial steady-state point \(E_0\) to be saddle-point stable, from lemma 1, the \((1 - \tau) f_k (k^*)\) schedule is steeper than the \(\delta (f (k^*) - g)\) schedule at \(E_0\).

Following Chamley (1981) and Epstein and Hynes (1983), suppose that the government raises capital tax \(\tau\) and pays back the revenue to consumers in a lump-sum manner, by keeping fiscal spending \(g\) constant. It shifts the \((1 - \tau) f_k (k^*)\) schedule downward, thereby bringing the steady-state point from point \(E_0\) to \(E_1\). Consequently, \(k^*\) decreases in response to the tax increase. Note that this reduction in \(k^*\) is larger than in the case of constant time preference: if \(\delta\) were constant, the reduction would stop at \(k'\). This property is contrasted sharply with the result of Epstein and Hynes (1983) under increasing marginal impatience, in which case the reduction in \(k^*\) caused by capital taxation is smaller than in the case of constant time preference. With
DMI, a decrease in $k^*$ makes consumers less patient, which raises the long-run after-tax interest rate irrespective of the capital taxation, and thereby causes a further reduction in $k^*$. From (21), in turn, the resulting decreases in steady-state consumption and welfare $u(c^*)/\delta(c^*)$ are larger than they would have been under constant time preference.

**Implication 1:** With DMI, an increase in capital taxes raises the long-run after-tax interest rate, so that, in contrast to the case of increasing marginal impatience, the resulting decreases in steady-state capital, consumption, and hence welfare are larger than they would have been under constant time preference.

### 3.3 The effects of government spending

By using (24) and hence Figure 4, we can also consider the effect of an increase in government spending, financed by lump-sum taxation. Suppose that the government increases its spending $g$ permanently by raising the lump-sum tax while keeping the capital tax $\tau$ constant. As shown in Figure 4, it shifts the $\delta(f(k) - g)$ schedule upward, thereby bringing the steady-state point from point $E_0$ to $E_2$ along the $(1 - \tau)f_k(k)$ schedule. An increase in $g$ thus raises $r^*$ and reduces $k^*$. The fiscal policy makes consumers cut down consumption and raises the rate of time preference under DMI, which leads to a higher $r^*$ and hence a smaller $k^*$. These properties differ from the result under constant time preference that either $r^*$ or $k^*$ is not affected by an increase in $g$, and the result under increasing marginal impatience that the $g$ increase lowers $r^*$ and thereby enlarges $k^*$.

**Implication 2:** With DMI, an increase in the government spending raises the long-run interest rate and harms capital accumulation.

### 4 Conclusions

This paper has demonstrated the implications of DMI for dynamic consumer behavior as well as for macroeconomic policy. We have first shown that with DMI, there are multiple steady-state nonsatiated and satiated equilibria. When the interest rate is constant, the nonsatiated steady-state point is necessarily unstable, which leads the rich to accumulate wealth up to a satiated level, and the poor to shrink toward zero consumption. In a capital
economy with decreasing returns technology, both of the nonsatiated and satiated steady-state points can be saddle-points stable. Unlike in the IMI case, the negative long-run effects of an increase in capital taxes on consumption, capital stocks, and hence welfare are larger than they would have been under constant time preference. An increase in government spending reduces the long-run capital stock.

There are interesting related issues. First, the model of constant interest rate economy can be applied straightforwardly to the analysis of various policy issues in small open economies. Second, in a two-country context, DMI leads to various interesting multiple steady-state equilibria, as discussed in Hirose and Ikeda (2004a). Third, the neoclassical model with DMI can be extended to a two-good framework, as in Hirose and Ikeda (2004b), where money is incorporated to examine the implications for the effect of inflation.
A Stability of points $E^*$ and $E^{**}$ in proposition 1

Local optimal dynamics around the nonsatiated steady-state point $(c^*, \phi^*, a^*)$ is linearized as

$$
\begin{pmatrix}
\dot{c} \\
\dot{\phi} \\
\dot{a}
\end{pmatrix}
= 
\begin{pmatrix}
0 & \frac{\delta \xi_m}{\xi_{cc}} & 0 \\
-\xi_c & r & 0 \\
-1 & 0 & r
\end{pmatrix}
\begin{pmatrix}
c - c^* \\
\phi - \phi^* \\
a - a^*
\end{pmatrix},
$$

where the coefficient matrix is evaluated at $(c^*, \phi^*, a^*)$. This system has three positive characteristic roots:

$$
\frac{1}{2}\left\{ r + \left( r^2 - \frac{4r\delta \xi_m}{\xi_{cc}} \right)^{1/2} \right\}, \quad \frac{1}{2}\left\{ r - \left( r^2 - \frac{4r\delta \xi_m}{\xi_{cc}} \right)^{1/2} \right\}, \quad \text{and} \quad r,
$$

implying that the nonsatiated steady-state point $(c^*, \phi^*, a^*)$ is unstable.

Local optimal dynamics around the satiated steady-state point $(c^{**}, \phi^{**}, a^{**})$ is linearized as

$$
\begin{pmatrix}
\dot{c} \\
\dot{\phi} \\
\dot{k}
\end{pmatrix}
= 
\begin{pmatrix}
\delta - r & \frac{\delta \xi_m}{\xi_{cc}} & 0 \\
0 & \delta & 0 \\
-1 & 0 & r
\end{pmatrix}
\begin{pmatrix}
c - c^{**} \\
\phi - \phi^{**} \\
a - a^{**}
\end{pmatrix},
$$

where the coefficient matrix is evaluated at $(c^{**}, \phi^{**}, a^{**})$. This system has characteristic roots $r$, $\delta(c^{**})$, and $\delta(c^{**}) - r$, which is negative since $\delta(c^{**}) < \delta(c^*) = r$ from the assumption (A6). The satiated steady-state point $(c^{**}, \phi^{**}, a^{**})$ is thus saddle-point stable.

B Proof of Lemma 1

By linearizing system (19) around the nonsatiated steady-state point $(c^*, \phi^*, k^*)$, the local dynamic system can be obtained as

$$
\begin{pmatrix}
\dot{c} \\
\dot{\phi} \\
\dot{k}
\end{pmatrix}
= 
\begin{pmatrix}
0 & \frac{\delta \xi_m}{\xi_{cc}} & -\frac{\xi_m(1-r)\delta k}{\xi_{cc}} \\
-\xi_c & \delta & 0 \\
-1 & 0 & f_k
\end{pmatrix}
\begin{pmatrix}
c - c^* \\
\phi - \phi^* \\
a - a^*
\end{pmatrix},
$$
where the coefficient matrix is evaluated at \((c^*, \phi^*, k^*)\). For this coefficient matrix,
\[
\text{trace} = \delta + f_k > 0, \quad \det. = \frac{\delta c}{\xi_{cc}} (\delta c f_k - (1 - \tau) f_{kk}).
\]
The linear system thus has two positive and one negative root if and only if
\[
\delta c (c^*) f_k (k^*) - (1 - \tau) f_{kk} (k^*) > 0,
\]
as stated as the first item in Lemma 1.

By linearizing the system (19) around the satiated steady-state point \((c^{**}, \phi^{**}, k^{**})\), the local dynamic system can be obtained as
\[
\begin{pmatrix}
\dot{c} \\
\dot{\phi} \\
\dot{k}
\end{pmatrix} =
\begin{pmatrix}
\delta - (1 - \tau) f_k & \frac{\delta c}{\xi_{cc}} & 0 \\
0 & \delta & 0 \\
-1 & 0 & f_k
\end{pmatrix}
\begin{pmatrix}
(c - c^{**}) \\
(\phi - \phi^{**}) \\
(a - a^{**})
\end{pmatrix},
\]
where the coefficient matrix is evaluated at \((c^{**}, \phi^{**}, k^{**})\). The characteristic roots are \(f_k\), \(\delta\), and \(\delta - (1 - \tau) f_k\). The linear system thus has two positive and one negative root if and only if \(\delta (c^{**}) - (1 - \tau) f_k (k^{**}) < 0\), as stated as the second statement in Lemma 1
References


Figure 1. Schedules of $u(c)/\delta(c)$ and $\delta(c)$
Figure 2. Optimal consumption dynamics under decreasing marginal impatience with a constant interest rate.
Figure 3. (a) Existence of non-satiated steady state equilibrium

when $\delta(c^*) > (1 - \tau)f_1(k^*)$.

Figure 3. (b) Existence of non-satiated steady state equilibrium

when $\delta(c^*) < (1 - \tau)f_1(k^*)$.
Figure 4. Steady-state capital stock under decreasing marginal impatience